Stat 851: Solutions to Assignment #3

(7.11) To see that A is a Borel set write A as

$$A = \{x_0\} = \{(-\infty, x_0) \cup (x_0, \infty)\}^c. \tag{$*$}$$

Since open intervals are Borel, so too are unions of open intervals, as are complements of unions of open intervals. Using $(*)$ and elementary properties of the Riemann integral, we have

$$\begin{aligned}
P(\{x_0\}) &= \int_{(-\infty,\infty)} \mathbb{1}_{\{x_0\}}(x) f(x)\,dx \\
&= \int_{(-\infty,x_0)} \mathbb{1}_{\{x_0\}}(x) f(x)\,dx + \int_{\{x_0\}} \mathbb{1}_{\{x_0\}}(x) f(x)\,dx + \int_{(x_0,\infty)} \mathbb{1}_{\{x_0\}}(x) f(x)\,dx \\
&= \int_{(-\infty,x_0)} 0 \cdot f(x)\,dx + \int_{\{x_0\}} 1 \cdot f(x)\,dx + \int_{(x_0,\infty)} 0 \cdot f(x)\,dx \\
&= 0 + \int_{x_0}^{x_0} 1 \cdot f(x)\,dx + 0 \\
&= 0
\end{aligned}$$

so that A is a null set for P.

(7.12) Suppose that B is countable. Enumerate the elements of B as $B = \{x_1, x_2, \ldots,\}$. Thus writing $B = \bigcup_{i=1}^{\infty} \{x_i\}$ expresses B as a disjoint union. Since P is a probability, we know that

$$P(B) = P\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} P(\{x_i\}).$$

But as proved in Exercise 7.11, $P(\{x_i\}) = 0$ for each i so that $P(B) = 0$ as well.

(7.13) If P and B are as in Exercise 7.12, and $P(A) = 1/2$, then since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ we conclude

$$\begin{aligned}
P(A \cup B) = \int_{-\infty}^{\infty} \mathbb{1}_{A\cup B}(x) f(x)\,dx &= \int_{-\infty}^{\infty} \mathbb{1}_{A}(x) f(x)\,dx + \int_{-\infty}^{\infty} \mathbb{1}_{B}(x) f(x)\,dx - \int_{-\infty}^{\infty} \mathbb{1}_{A\cap B}(x) f(x)\,dx \\
&= 1/2 + 0 - \int_{-\infty}^{\infty} \mathbb{1}_{A\cap B}(x) f(x)\,dx.
\end{aligned}$$

We know from Exercise 7.12 that B is a null set for P (and that B is actually a Borel set). Since we are told that A is an event, we can conclude that $A \cap B$ is an event as well. Since $A \cap B \subseteq B$, we see that $P(A \cap B) \leq P(B) = 0$ so that

$$\int_{-\infty}^{\infty} \mathbb{1}_{A\cap B}(x) f(x)\,dx = 0,$$

and therefore $P(A \cup B) = 1/2$, as required.

An alternative solution is as follows. Since B as given by Exercise 7.12 is a Borel set, and since A is assumed to be an event, we know that $A \cup B$ is also an event. It now follows that $P(A \cup B) = 1/2$ since $A \subseteq B$ implies

$$\frac{1}{2} = P(A) \le P(A \cup B) \le P(A) + P(B) = \frac{1}{2} + 0 = \frac{1}{2}.$$

(7.14) Suppose that $A_1, A_2, \dots$ is a sequence of null sets. This means that there exist sets $C_1, C_2, \dots$ with $A_i \subset C_i$ and $P(C_i) = 0$ for each i. Let

$$C = \bigcup_{i=1}^{\infty} C_i$$

so that

$$B = \bigcup_{i=1}^{\infty} A_i \subset C.$$

Since

$$P(C) = P\left(\bigcup_{i=1}^{\infty} C_i\right) \le \sum_{i=1}^{\infty} P(C_i) = 0$$

we conclude that B is a null set for P.

(7.15) Suppose that $E(|X|) = 0$. To show that $X = 0$ except possibly on a null set means to show that $P(X = 0) = 1$. We will prove $P(X = 0) = 1$ by deriving a contradiction. Suppose, to the contrary, that $P(X = 0) < 1$. Then, there exists some $a > 0$ such that $P(|X| \ge a) > 0$. However, by Markov's inequality (Corollary 5.1), we have that for every $a > 0$,

$$P(|X| \ge a) \le \frac{E(|X|)}{a} = 0$$

since $E(|X|) = 0$ by assumption. Hence, for every $a > 0$, we have $P(|X| \ge a) = 0$, and we conclude $P(|X| > 0) = 0$, or in other words, $P(X = 0) = 1$.

It is not possible to conclude in general that $X = 0$ everywhere. As a simple example, suppose that $\Omega = \{0, 1\}$ and let P be the Dirac mass at the point 0. (See Example 2 on page 42.) It then follows that the random variable $X : \Omega \to \{0, 1\}$ whose law (or distribution) is P has $P(X = 0) = 1$ and $P(X = 1) = 0$ so that $E(|X|) = 0$, even though $X \ne 0$ everywhere (i.e., $X(\omega) \ne 0$ for some $\omega \in \Omega$).

(7.17) A direct application of Corollary 7.1 gives

(a) $P\left(\left(-\frac{1}{2}, \frac{1}{2}\right) \right) = F\left(\frac{1}{2}-\right) - F\left(-\frac{1}{2}\right) = 1/4 - 0 = 1/4,$

(b) $P\left(\left(-\frac{1}{2}, \frac{3}{2}\right) \right) = F\left(\frac{3}{2}-\right) - F\left(-\frac{1}{2}\right) = (1/4 + 1/2) - 0 = 3/4,$

(c) $P\left(\left(\frac{2}{3}, \frac{5}{2}\right) \right) = F\left(\frac{5}{2}-\right) - F\left(\frac{2}{3}\right) = (1/4 + 1/2 + 1/4) - (1/4) = 3/4,$

(d) $P\left([0, 2) \right) = F(2-) - F(0-) = (1/4 + 1/2) - 0 = 3/4,$

(e) $P\left((3, \infty) \right) = 1 - P\left((-\infty, 3] \right) = 1 - F(3) = 1 - (1/4 + 1/2 + 1/4) = 0.$

(7.18) In order to prove that the function F given by

$$F(x) = \sum_{i=1}^{\infty} 2^{-i} \mathbb{1}_{[\frac{1}{i},\infty)}(x)$$

is a distribution function of a probability on $\mathbb{R}$ we use Theorem 7.2. Clearly, $F(x) = 0$ for all $x \le 0$, so that

$$\lim_{x\to-\infty} F(x) = 0.$$

Moreover, for all $x \ge 1$,

$$F(x) = \sum_{i=1}^{\infty} 2^{-i} \mathbb{1}_{[\frac{1}{i},\infty)}(x) = \sum_{i=1}^{\infty} 2^{-i} = 1$$

so that

$$\lim_{x\to\infty} F(x) = 1.$$

Suppose that $0 < x < y$. If $x \ge 1/i$ for some i, then necessarily $y > 1/i$ since $y > x$. In particular, $\mathbb{1}_{[\frac{1}{i},\infty)}(x) \le \mathbb{1}_{[\frac{1}{i},\infty)}(y)$ for all $-\infty < x < y < \infty$ so that F is non-decreasing. We have already shown that $F(x) = 0$ for all $x \le 0$ and that $F(x) = 1$ for all $x \ge 1$ so that F is necessarily right continuous on $(-\infty, 0) \cup [1, \infty)$. We must still show that F is right continuous for all $x \in [0, 1)$. Notice that F is a step function for $0 \le x < 1$ with jumps at the points $x = 1/i$, $i = 2, 3, \ldots$. It is therefore clear that F is continuous on each open interval $((i+1)^{-1}, i^{-1})$, for $i = 1, 2, \ldots$. Suppose that $x = 1/i$ for some $i = 1, 2, \ldots$. Then, for all y with $1/(i-1) > y > 1/i$ we have $F(y) = F(1/i)$ so that

$$F(x+) = F(1/i+) = \lim_{y\to 1/i+} F(y) = \lim_{y\to 1/i+,\, y>1/(i-1)} F(1/i) = F(1/i).$$

It remains to show that F is right continuous at 0; that is, we must show

$$F(0+) = \lim_{y\to 0+} F(y) = 0. \tag{$\dagger$}$$

To prove (†), we show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(y) < \varepsilon$ whenever $y < \delta$. Let $\varepsilon > 0$ be arbitrary. Then there exists an $i_0 \in \mathbb{N}$ such that $\varepsilon \ge 2^{-i_0}$. Let $\delta = 1/i_0$ so that $y < 1/i_0$. Thus, by the right-continuity of F,

$$F(y) \le F(1/i_0) = \sum_{i=1}^{\infty} 2^{-i} \mathbb{1}_{[\frac{1}{i},\infty)}(1/i_0) = \sum_{i=i_0}^{\infty} 2^{-i} = 1 - \sum_{i=1}^{i_0-1} 2^{-i} = 1 - \frac{1}{2}\cdot\frac{1-2^{-i_0}}{1-1/2} = 2^{-i_0} \le \varepsilon.$$

Thus, by Theorem 7.2, F is the distribution function of a probability on $\mathbb{R}$.

Finally, a direct application of Corollary 7.1 gives

(a) $P(\,[1,\infty)\,) = 1 - F(1-) = 1 - \sum_{i=2}^{\infty} 2^{-i} \mathbb{1}_{[\frac{1}{i},\infty)} = 1 - \sum_{i=2}^{\infty} 2^{-i} = 1 - 1/2 = 1/2,$

(b) $P\left(\left[\frac{1}{10},\infty\right)\right) = 1 - F\left(\frac{1}{10}-\right) = 1 - \sum_{i=11}^{\infty} 2^{-i} \mathbb{1}_{[\frac{1}{i},\infty)} = \sum_{i=1}^{10} 2^{-i} = \frac{1-2^{-11}}{1-1/2} - \frac{1}{2} = 1 - 2^{-10},$

(c) $P(\,\{0\}\,) = F(0) - F(0-) = 0 - 0 = 0,$

(d) $P\left(\left[0,\frac{1}{2}\right)\right) = F\left(\frac{1}{2}-\right) - F\,(0-) = \sum_{i=3}^{\infty} 2^{-i}\mathbb{1}_{\left[\frac{1}{i},\infty\right)} - 0 = 1 - \sum_{i=1}^{2} 2^{-i} = 1-(1/2+1/4) = 1/4,$

(e) $P\,(\,(-\infty,0)\,) = F(0-) - 0 = 0,$

(f) $P\,(\,(0,\infty)\,) = 1 - P\,(\,(-\infty,0]\,) = 1 - F(0) = 1 - 0 = 1.$