Stat 451: Solutions to Assignment #2

(3.1) If $A \cap B = \emptyset$, then by Theorem 2.2 we conclude $P(A \cap B) = P(\emptyset) = 0$. Hence, in order for $A$ and $B$ to be independent, it must be the case that $P(A \cap B) = P(A) \cdot P(B) = 0$. The product of two real numbers is 0 if and only if at least one of those numbers is 0. We thus conclude that at least one of $P(A)$ and $P(B)$ must be 0 in order for $A$ and $B$ to be independent.

(3.3) Suppose $(\Omega, \mathcal{A}, P)$ is a probability space and that $C \in \mathcal{A}$ with $P(C) > 0$. If $Q(A) = P(A|C)$ for $A \in \mathcal{A}$, then by Theorem 3.2 (b) it follows that $Q$ is a probability measure on $(\Omega, \mathcal{A})$. It now follows from Theorem 2.2 that $Q$ is additive. That is, if $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$P\left(\bigcup_{i=1}^{n} A_i \middle| C\right) = Q\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Q(A_i) = \sum_{i=1}^{n} P(A_i|C).$$

(3.6) Using the definition of conditional probability,

$$P(\text{you have AIDS}|\text{test positive}) = \frac{P(\text{test positive}|\text{you have AIDS}) \cdot P(\text{you have AIDS})}{P(\text{test positive})}.$$

We now use the information given in the problem, but need to be careful about the wording. We are told that $P(\text{you have AIDS}) = 1/10000 = 0.0001$, and $P(\text{test positive}|\text{you have AIDS}) = 0.99$. However, the 5% false positive means $P(\text{test positive}|\text{you do NOT have AIDS}) = 0.05$. Therefore, we must calculate $P(\text{test positive})$ using Exercise 3.5. Thus,

$$\begin{aligned}
&P(\text{test positive}) \\
&\quad = P(\text{test positive}|\text{you have AIDS}) \cdot P(\text{you have AIDS}) \\
&\qquad + P(\text{test positive}|\text{you do NOT have AIDS}) \cdot P(\text{you do NOT have AIDS}) \\
&\quad = 0.99 \cdot 0.0001 + 0.05 \cdot 0.9999 \\
&\quad = 0.050094
\end{aligned}$$

so that

$$P(\text{you have AIDS}|\text{test positive}) = \frac{0.99 \times 0.0001}{0.050094} = \frac{1}{506} \approx 0.001976.$$

(3.11) Suppose that $R_i$ is the event {red ball on draw $i$}, and that $B_i$ is the event {blue ball on draw $i$}. The problem specifies that

$$P(B_1) = \frac{b}{b+r}, \quad P(R_1) = \frac{r}{b+r}, \quad P(B_2|B_1) = \frac{b+d}{b+r+d}, \quad P(B_2|R_1) = \frac{b}{b+r+d}.$$

(a) Hence, using Exercise 3.5, we conclude that

$$P(B_2) = P(B_2|B_1) \cdot P(B_1) + P(B_2|R_1) \cdot P(R_1) = \frac{b+d}{b+r+d} \cdot \frac{b}{b+r} + \frac{b}{b+r+d} \cdot \frac{r}{b+r} = \frac{b}{b+r}.$$

(b) It then follows from Bayes' Theorem that

$$P(B_1|B_2) = \frac{P(B_2|B_1) \cdot P(B_1)}{P(B_2)} = \frac{\dfrac{b+d}{b+r+d} \cdot \dfrac{b}{b+r}}{\dfrac{b}{b+r}} = \frac{b+d}{b+r+d}.$$

(3.12) Let $B_n$ denote the event that the $n$th ball drawn is blue. We will prove by induction that $P(B_n) = P(B_1)$ for all $n \geq 1$. When $n = 2$, the direct computation in Exercise 3.11 shows $P(B_2) = P(B_1)$. Suppose now that $P(B_N) = P(B_1)$ for some $N \geq 1$. We will show that $P(B_{N+1}) = P(B_1)$. Assume that at time $N$ there are $r'$ red balls and $b'$ blue balls in the urn. Thus,

$$P(B_N) = \frac{b'}{r' + b'}.$$

But, by the induction hypothesis, $P(B_N) = P(B_1)$ so that

$$P(B_N) = \frac{b'}{r' + b'} = \frac{b}{r + b}. \tag{H}$$

Similar,

$$P(R_N) = \frac{r'}{r' + b'} = \frac{r}{r + b}.$$

It follows from Exercise 3.5 that

$$\begin{aligned} P(B_{N+1}) &= P(B_{N+1}|B_N) \cdot P(B_N) + P(B_{N+1}|R_N) \cdot P(R_N) \\ &= \frac{b' + d}{r' + b' + d} \cdot \frac{b'}{r' + b'} + \frac{b'}{r' + b' + d} \cdot \frac{r'}{r' + b'} \\ &= \frac{b'}{r' + b'} \\ &= \frac{b}{r + b} = P(B_1) \text{ by the induction hypothesis (H)} \end{aligned}$$

Thus, by induction, $P(B_n) = P(B_1)$ for all $n \geq 1$.

(3.13) We must compute $P(B_1|B_2 \cap \cdots \cap B_{n+1})$. By definition of conditional probability,

$$P(B_1|B_2 \cap \cdots \cap B_{n+1}) = \frac{P(B_1 \cap B_2 \cap \cdots \cap B_{n+1})}{P(B_2 \cap \cdots \cap B_{n+1})}. \tag{$*$}$$

Using Theorem 3.3, we calculate

$$\begin{aligned} P(B_1 \cap \cdots \cap B_{n+1}) &= P(B_1) \cdot P(B_2|B_1) \cdot P(B_3|B_1 \cap B_2) \cdot \cdots \cdot P(B_{n+1}|B_1 \cap \cdots \cap B_n) \\ &= \frac{b}{b + r} \cdot \frac{b + d}{b + r + d} \cdot \frac{b + 2d}{b + r + 2d} \cdot \cdots \cdot \frac{b + nd}{b + r + nd} \\ &= \prod_{k=0}^{n} \frac{b + kd}{b + r + kd}. \end{aligned} \tag{$\dagger$}$$

Using Exercise 3.5, we find

$$P(B_2 \cap \cdots \cap B_{n+1}) = P(B_1 \cap B_2 \cap \cdots \cap B_{n+1}) + P(R_1 \cap B_2 \cap \cdots \cap B_{n+1}). \tag{$**$}$$

We can again use Theorem 3.3 to find that

$$\begin{aligned} P(R_1 \cap B_2 \cap \cdots \cap B_{n+1}) &= P(R_1) \cdot P(B_2|R_1) \cdot P(B_3|R_1 \cap B_2) \cdot \cdots \cdot P(B_{n+1}|R_1 \cap B_2 \cap \cdots \cap B_n) \\ &= \frac{r}{b + r} \cdot \frac{b}{b + r + d} \cdot \frac{b + d}{b + r + 2d} \cdot \cdots \cdot \frac{b + (n-1)d}{b + r + nd} \\ &= \frac{r}{b + r} \prod_{k=1}^{n} \frac{b + (k-1)d}{b + r + kd}. \end{aligned} \tag{$\ddagger$}$$

Substituting (†) and (‡) into (∗∗) yields

$$P(B_2 \cap \cdots \cap B_{n+1}) = \prod_{k=0}^{n} \frac{b+kd}{b+r+kd} + \frac{r}{b+r} \prod_{k=1}^{n} \frac{b+(k-1)d}{b+r+kd}. \qquad (***)$$

Finally, substituting (∗ ∗ ∗) and (†) into (∗) gives

$$\begin{aligned} P(B_1|B_2 \cap \cdots \cap B_{n+1}) &= \frac{\prod_{k=0}^{n} \frac{b+kd}{b+r+kd}}{\prod_{k=0}^{n} \frac{b+kd}{b+r+kd} + \frac{r}{b+r} \prod_{k=1}^{n} \frac{b+(k-1)d}{b+r+kd}} \\ &= \frac{1}{1 + \frac{r}{r+b} \cdot \frac{b+r}{b+nd}} \\ &= \frac{b+nd}{b+r+nd}. \end{aligned}$$

Note that

$$\lim_{n\to\infty} P(B_1|B_2 \cap \cdots \cap B_{n+1}) = \lim_{n\to\infty} \frac{b+nd}{b+r+nd} = 1.$$

(4.1) If $P$ is the binomial$(n, p)$ distribution, then

$$P(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!}\, p^k (1-p)^{n-k}.$$

Substituting $\lambda = pn$ gives

$$p^k(1-p)^{n-k} = \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \lambda^k n^{-k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}.$$

Furthermore,

$$\frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$$

so that combining everything gives

$$\begin{aligned} P(k \text{ successes}) &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \lambda^k n^{-k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left\{ \frac{n \cdot (n-1) \cdots (n-k+1)}{n^k} \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left\{ \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \end{aligned}$$

Next, taking the limit as $n \to \infty$, $\lambda$ =constant, gives

$$
\begin{aligned}
&\lim P(k \text{ successes}) \\
&\quad = \lim \left[ \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left\{ \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \right] \\
&\quad = \lim \left[\frac{\lambda^k}{k!}\right] \lim \left[\left(1 - \frac{\lambda}{n}\right)^n\right] \lim \left[\left\{ \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \right\}\right] \lim \left[\left(1 - \frac{\lambda}{n}\right)^{-k}\right] \\
&\quad = \frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot 1 \cdot 1 \\
&\quad = \frac{e^{-\lambda}\lambda^k}{k!}
\end{aligned}
$$