Stat 451: Solutions to Assignment #1

(2.1) By definition, 2^Ω is the set of all subsets of Ω. Therefore, to show that 2^Ω is a σ-algebra we must show that the conditions of the definition σ-algebra are met. In particular,

- $\emptyset \subseteq \Omega$ so that $\emptyset \in 2^\Omega$,
- $\Omega \subseteq \Omega$ so that $\Omega \in 2^\Omega$,
- if $A \subseteq \Omega$, then $A^c = A^c \cap \Omega \subseteq \Omega$ so that $A^c \in 2^\Omega$, and
- if $A_1, A_2, \ldots \in 2^\Omega$ so that each $A_i \subseteq \Omega$, then necessarily

$$\bigcup_{i=1}^{\infty} A_i \subseteq \Omega \quad \text{implying} \quad \bigcup_{i=1}^{\infty} A_i \in 2^\Omega.$$

Notice that this proof works whether or not Ω is finite, countable, or uncountable.

In the case when Ω is finite, we also have $|2^\Omega| = 2^{|\Omega|} < \infty$. To prove this fact, suppose that $|\Omega| = N$. The number of k element subsets of Ω is $\binom{N}{k}$ for $k = 0, 1, \ldots, N$. Thus, the total number of subsets of Ω is

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N < \infty.$$

In other words, $|2^\Omega| = 2^{|\Omega|}$ which shows that the power set of a finite set is necessarily finite.

(2.2) Since $\mathcal{G}_\alpha$ is a σ-algebra for each α, it follows that $\emptyset \in \mathcal{G}_\alpha$ for each α. Hence, $\emptyset \in \cap_\alpha \mathcal{G}_\alpha = \mathcal{H}$. Similarly, $\Omega \in \mathcal{G}_\alpha$ for each α since $\mathcal{G}_\alpha$ is a σ-algebra. Thus, $\Omega \in \cap_\alpha \mathcal{G}_\alpha = \mathcal{H}$. Suppose that $A \in \mathcal{H}$, so that $A \in \mathcal{G}_\alpha$ for each α. Since each $\mathcal{G}_\alpha$ is a σ-algebra, each contains A^c. That is, $A^c \in \mathcal{G}_\alpha$ for each α so that $A^c \in \cap_\alpha \mathcal{G}_\alpha = \mathcal{H}$. Finally, suppose that $A_1, A_2, \ldots \in \mathcal{H}$. It follows that $A_1, A_2, \ldots \in \mathcal{G}_\alpha$ for each α so that

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}_\alpha \quad \text{for each } \alpha$$

since $\mathcal{G}_\alpha$ is a σ-algebra. Hence,

$$\bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha} \mathcal{G}_\alpha = \mathcal{H}.$$

(2.3) (a) Recall that to show that events E and F are equal, $E = F$, it suffices to show that both $E \subseteq F$ and $F \subseteq E$. Now, let

$$x \in \left(\bigcup_{n=1}^{\infty} A_n \right)^c \quad \text{so that} \quad x \notin \bigcup_{n=1}^{\infty} A_n.$$

But this means that $x \notin A_1$, and $x \notin A_2$, and $\ldots$, or equivalently, $x \in A_1^c$, and $x \in A_2^c$, and $\ldots$. Hence,

$$x \in \bigcap_{n=1}^{\infty} A_n^c.$$

Conversely, suppose that

$$x \in \bigcap_{n=1}^{\infty} A_n^c$$

so that $x \notin A_n$ for each n. But this means that $x \notin \bigcup_{n=1}^{\infty} A_n$, or equivalently,

$$x \in \left(\bigcup_{n=1}^{\infty} A_n \right)^c.$$

Having shown both containments, DeMorgan's first law follows.

(b) The second part now follows immediately from the first part. Let $B_n = A_n^c$, so that the first part yields

$$\left(\bigcup_{n=1}^{\infty} B_n^c \right)^c = \bigcap_{n=1}^{\infty} B_n.$$

Taking complements of both sides gives

$$\bigcup_{n=1}^{\infty} B_n^c = \left(\bigcap_{n=1}^{\infty} B_n \right)^c.$$

However, the labels A_n and B_n are arbitrary; whence the result.

(2.6) Suppose that $B \in \mathcal{A}$, and let $\mathcal{F} = \{A \cap B : A \in \mathcal{A}\}$. To show $\mathcal{F}$ is a σ-algebra of subsets of B requires that we show

- $\emptyset \in \mathcal{F}$, $B \in \mathcal{F}$,
- $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$ (where F^c means the complement of F in B, namely $B \setminus F$) , and
- $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ whenever $F_i \in \mathcal{F}$ for each i.

Since $\mathcal{A}$ is a σ-algebra, we know that $\emptyset \in \mathcal{A}$. Thus, we find that $\emptyset \cap B = \emptyset \in \mathcal{F}$. Similarly, since $\Omega \in \mathcal{A}$, we find that $\Omega \cap B = B \in \mathcal{F}$. Suppose that $F \in \mathcal{F}$. Then $F = A \cap B$ for some $A \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, we know that $A^c \in \mathcal{A}$. Therefore,

$$F^c = B \setminus F = B \cap (A \cap B)^c = B \cap (A^c \cup B^c) = (B \cap A^c) \cup (B \cap B^c) = B \cap A^c \in \mathcal{F}.$$

Suppose that $F_i \in \mathcal{F}$. Then $F_i = A_i \cap B$ for some $A_i \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, we know that $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$. Therefore,

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (A_i \cap B) = B \cap \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Notice that nothing in the proof required that $B \in \mathcal{A}$. Therefore, even if $B \subset \Omega$ with $B \notin \mathcal{A}$, we still have that $\mathcal{F}$ is a σ-algebra.

(2.9) This follows immediately from countable additivity (Definition 2.3, axiom 2). Simply let $A_1 = A$, $A_2 = B$, and $A_n = \emptyset$ for all $n \geq 3$. Indeed this fact is simply an application of Theorem 2.2 (ii).

(2.10) Since $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$ and $(A \cap B^c) \cap (A \cap B) = (A \cap B) \cap (A^c \cap B) = (A \cap B^c) \cap (A^c \cap B) = \emptyset$, we can apply Exercise 2.9 to conclude

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B). \tag{$*$}$$

However, we can also write $A \cup B = A \cup (A^c \cap B)$. Since $A \cap (A^c \cap B) = \emptyset$, we can again use Exercise 2.9 to conclude

$$P(A \cup B) = P(A) + P(A^c \cap B) \text{ or } P(A^c \cap B) = P(A \cup B) - P(A). \tag{$\dagger$}$$

Finally, we can also write $A \cup B = B \cup (A \cap B^c)$. Since $B \cap (A \cap B^c) = \emptyset$, a third application of Exercise 2.9 gives

$$P(A \cup B) = P(B) + P(A \cap B^c) \text{ or } P(A \cap B^c) = P(A \cup B) - P(B). \tag{$\ddagger$}$$

Substituting the expressions (†) and (‡) into the expression (∗) gives

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) = P(A \cup B) - P(B) + P(A \cap B) + P(A \cup B) - P(A).$$

Solving gives

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

as required.

(2.11) Since $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$, we can apply Exercise 2.9 to conclude $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$. Thus, $P(A) = 1 - P(A^c)$ as required.

(2.12) Note that we proved this fact in our solution to Exercise 2.10. To repeat, since $(A \cap B^c) \cup (A \cap B) = A$ and $(A \cap B^c) \cap (A \cap B) = \emptyset$, an application of Exercise 2.9 gives $P(A) = P(A \cap B^c) + P(A \cap B)$. Hence, $P(A \cap B^c) = P(A) - P(A \cap B)$ as required.

(2.14) Since $A \cap B \subset B$, it follows that $P(A \cap B) \leq P(B) = \frac{1}{3}$. (See page 9.) Since $P(A \cup B) \leq 1$, we conclude from Exercise 2.10 that $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = \frac{3}{4} + \frac{1}{3} - 1 = \frac{1}{12}$. Together these imply $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$.

(2.15) Define a new sequence of sets (B_n) as follows. Let $B_1 = A_1$, $B_2 = A_2 \setminus B_1$, and for $n \geq 3$,

$$B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} B_i \right).$$

Then, by construction, the (B_n) are pairwise disjoint, and

$$\bigcup_{m=1}^{n} B_m = \bigcup_{m=1}^{n} A_m.$$

Hence, using the finite additivity of the probability measure P, we conclude

$$P\left(\bigcup_{m=1}^{n} A_m \right) = P\left(\bigcup_{m=1}^{n} B_m \right) = \sum_{m=1}^{n} P(B_m).$$

But, $B_n \subseteq A_n$ for each n, so that $P(B_n) \leq P(A_n)$. Hence,

$$\sum_{m=1}^{n} P(B_m) \leq \sum_{m=1}^{n} P(A_m),$$

establishing the result.

(2.17) Since $\emptyset$ is by convention finite, $\emptyset \in \mathcal{A}$. It then follows that $\Omega \in \mathcal{A}$ since Ω has a finite complement, namely $\Omega^c = \emptyset$. Suppose that $A \in \mathcal{A}$. Then there are two possibilities. Either A is a finite set, or A has a finite complement. Suppose first that A is a finite set. Then A^c must have a finite complement (namely $A = (A^c)^c$). Thus, $A^c \in \mathcal{A}$. Conversely, suppose that A has a finite complement. Then A^c is finite and so $A^c \in \mathcal{A}$. In either case, $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$. Next, suppose that $A_1, A_2, \ldots, A_n \in \mathcal{A}$. Then each A_i is either finite or has finite complement. If all the A_i are finite, then obviously $\cup_{i=1}^{n} A_i$ is finite so that $\cup_{i=1}^{n} A_i \in \mathcal{A}$. If, on the other hand, none of the A_i are finite, but each has finite complement, then

$$\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i \right)^c$$

is finite. (That is, a finite intersection of finite sets must be finite.) But it then follows that $\cup_{i=1}^{n} A_i \in \mathcal{A}$ since $\cup_{i=1}^{n} A_i$ has finite complement. Now, suppose that there is at least one finite set, and at least one set with finite complement. Without loss of generality, suppose that the finite sets are $A_1, \ldots, A_j$, and the sets with finite complement are $A_{j+1}, \ldots, A_n$. (We are assuming that $1 \leq j < n$.) Since $A_{j+1}^c, \ldots, A_n^c$ are each finite, we have as above that

$$\bigcap_{i=j+1}^{n} A_i^c$$

is finite. Since the intersection of a finite set with any other set is necessarily finite, we conclude that

$$\bigcap_{i=1}^{j} A_i^c \cap \bigcap_{i=j+1}^{n} A_i^c = \bigcap_{i}^{n} A_i^c = \left(\bigcup_{i=j+1}^{n} A_i \right)^c$$

must be finite. Thus, $\cup_{i=1}^{n} A_i$ has finite complement so that

$$\bigcup_{i=1}^{n} A_i \in \mathcal{A}.$$

Hence, $\mathcal{A}$ is an algebra.

We now show that $\mathcal{A}$ is not a σ-algebra. Suppose that Ω is an infinite set. Let A be a countable subset of Ω such that A^c is infinite. (We prove below that it is always possible to find such a set.) Since A is countable, it is possible to enumerate its members; that is, $A = \{x_1, x_2, \ldots\}$. Now, set $A_i = \{x_i\}$ for $i = 1, 2, \ldots$. Then each $A_i \in \mathcal{A}$ since A_i is finite, and $A_i \cap A_j = \emptyset$ for every $i \neq j$. However,

$$\bigcup_{i=1}^{\infty} A_i = \{x_1, x_2, \ldots\} = A \notin \mathcal{A}$$

since A is assumed countably infinite, and since A^c is also assumed infinite.

Lemma. If Ω is an infinite set, then there exists $A \subset \Omega$ such that A is countable and A^c is infinite.

Proof. It follows from the Axiom of Choice that every uncountable set contains a countable subset. Hence, if Ω is uncountable and A is a countable subset, then $\Omega \setminus A$ is also uncountable (i.e., A^c is infinite). Suppose, on the other hand, that Ω is countable. Enumerating its elements gives $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$. Set $A = \{\omega_1, \omega_3, \omega_5, \ldots\}$ so that both A and $A^c = \{\omega_2, \omega_4, \omega_6, \ldots\}$ are countable.