Statistics 441 Winter 2009 Midterm – Solutions

1. (a) Since $B_t \sim \mathcal{N}(0,t)$, we find $\mathbb{E}(B_4^2 + 3B_4 + 2) = \mathbb{E}(B_4^2) + 3\mathbb{E}(B_4) + 2 = 4 + 0 + 2 = 6$.

1. (b) Since $\{B_t^2 - t, t \geq 0\}$ is a martingale, we find $\mathbb{E}(B_3^2 - 3|\mathcal{F}_2) = B_2^2 - 2$, so that

$$\mathbb{E}(B_3^2|\mathcal{F}_2) = B_2^2 + 1.$$

1. (c) This expression represents the quadratic variation of Brownian motion on $[0,2]$. Thus,

$$\lim_{n\to\infty} \sum_{j=1}^{2n} (B_{j/n} - B_{(j-1)/n})^2 = 2.$$

Informally, we can write

$$\lim_{n\to\infty} \sum_{j=1}^{2n} (B_{j/n} - B_{(j-1)/n})^2 = \int_0^2 (\mathrm{d}B_s)^2 = \int_0^2 \mathrm{d}s = 2.$$

2. (a) We find

$$\int_0^2 \sqrt{s}\, \mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^2 s\, \mathrm{d}s\right) = \mathcal{N}(0,2).$$

2. (b) We know

$$\mathrm{var}\left(\int_0^1 e^{B_s}\, \mathrm{d}B_s\right) = \int_0^1 \mathbb{E}(e^{2B_s})\, \mathrm{d}s.$$

Since $B_s \sim \mathcal{N}(0,s)$, we can write $\mathbb{E}(e^{2B_s}) = \mathbb{E}(e^{2\sqrt{s}\,Z})$ for $Z \sim \mathcal{N}(0,1)$. We recognize this as the moment generating function of Z evaluated at $2\sqrt{s}$. Since $\mathbb{E}(e^{\theta Z}) = e^{\theta^2/2}$, we conclude $\mathbb{E}(e^{2B_s}) = e^{4s/2} = e^{2s}$. Therefore,

$$\mathrm{var}\left(\int_0^1 e^{B_s}\, \mathrm{d}B_s\right) = \int_0^1 \mathbb{E}(e^{2B_s})\, \mathrm{d}s = \int_0^1 e^{2s}\, \mathrm{d}s = \frac{e^2 - 1}{2}.$$

3. (a) If $f(x) = x^3$ so that $f'(x) = 3x^2$ and $f''(x) = 6x$, then Itô's formula implies

$$\mathrm{d}f(B_t) = f'(B_t)\, \mathrm{d}B_t + \frac{1}{2} f''(B_t)\, \mathrm{d}t,$$

and so if $X_t = B_t^3$, then

$$\mathrm{d}X_t = 3B_t^2\, \mathrm{d}B_t + 3B_t\, \mathrm{d}t = 3X_t^{2/3}\, \mathrm{d}B_t + 3X_t^{1/3}\, \mathrm{d}t.$$

3. (b) If $f(t,x) = \sqrt{t}\, x$ so that $f'(t,x) = \sqrt{t}$, $f''(x) = 0$, and $\dot{f}(t,x) = \frac{x}{2\sqrt{t}}$, then Itô's formula implies

$$\mathrm{d}f(t,B_t) = f'(t,B_t)\, \mathrm{d}B_t + \frac{1}{2} f''(t,B_t)\, \mathrm{d}t + \dot{f}(t,B_t)\, \mathrm{d}t,$$

and so if $X_t = \sqrt{t}\, B_t$, then

$$\mathrm{d}X_t = \sqrt{t}\, \mathrm{d}B_t + \frac{B_t}{2\sqrt{t}}\, \mathrm{d}t = \sqrt{t}\, \mathrm{d}B_t + \frac{X_t}{2t}\, \mathrm{d}t.$$

3. (c) If $f(x) = x^2$ so that $f'(x) = 2x$ and $f''(x) = 2$, then Itô's formula implies

$$\mathrm{d}f(Y_t) = f'(Y_t)\,\mathrm{d}Y_t + \frac{1}{2}f''(Y_t)\,\mathrm{d}\langle Y\rangle_t.$$

Since $\mathrm{d}Y_t = 2\sqrt{Y_t}\,\mathrm{d}B_t + 3\,\mathrm{d}t$ we see that $\mathrm{d}\langle Y\rangle_t = 4Y_t\,\mathrm{d}t$, and so if $X_t = Y_t^2$, then

$$\begin{aligned}
\mathrm{d}X_t &= 2Y_t\,\mathrm{d}Y_t + \mathrm{d}\langle Y\rangle_t \\
&= 2Y_t\left[2\sqrt{Y_t}\,\mathrm{d}B_t + 3\,\mathrm{d}t\right] + 4Y_t\,\mathrm{d}t \\
&= 4Y_t^{3/2}\,\mathrm{d}B_t + 6Y_t\,\mathrm{d}t + 4Y_t\,\mathrm{d}t \\
&= 4Y_t^{3/2}\,\mathrm{d}B_t + 10Y_t\,\mathrm{d}t \\
&= 4X_t^{3/4}\,\mathrm{d}B_t + 10\sqrt{X_t}\,\mathrm{d}t.
\end{aligned}$$

4. If

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{\mathrm{d}B_s}{1-s},$$

then the chain rule implies

$$\begin{aligned}
\mathrm{d}X_t &= -a\,\mathrm{d}t + b\,\mathrm{d}t + (1-t)\,\mathrm{d}\left(\int_0^t \frac{\mathrm{d}B_s}{1-s}\right) + \left(\int_0^t \frac{\mathrm{d}B_s}{1-s}\right)\mathrm{d}(1-t) \\
&= -a\,\mathrm{d}t + b\,\mathrm{d}t + (1-t)\cdot\frac{\mathrm{d}B_t}{1-t} - \left(\int_0^t \frac{\mathrm{d}B_s}{1-s}\right)\mathrm{d}t \\
&= -a\,\mathrm{d}t + b\,\mathrm{d}t + \mathrm{d}B_t - \left(\int_0^t \frac{\mathrm{d}B_s}{1-s}\right)\mathrm{d}t \\
&= \left[b - a - \int_0^t \frac{\mathrm{d}B_s}{1-s}\right]\mathrm{d}t + \mathrm{d}B_t.
\end{aligned}$$

Writing

$$\int_0^t \frac{\mathrm{d}B_s}{1-s} = \frac{X_t - a(1-t) - bt}{1-t}$$

therefore implies

$$\mathrm{d}X_t = \left[b - a - \int_0^t \frac{\mathrm{d}B_s}{1-s}\right]\mathrm{d}t + \mathrm{d}B_t = \left[b - a - \frac{X_t - a(1-t) - bt}{1-t}\right]\mathrm{d}t + \mathrm{d}B_t$$

which upon simplification reduces to

$$\mathrm{d}X_t = \frac{b - X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t$$

as required.

5. As $\sigma \to \infty$, we see that

$$\frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\frac{\log(S_0/E)+rT}{\sigma} + \frac{T\sigma}{2}}{\sqrt{T}} \to \infty$$

and

$$\frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\frac{\log(S_0/E)+rT}{\sigma} - \frac{T\sigma}{2}}{\sqrt{T}} \to -\infty.$$

Thus,

$$V(0, S_0) \to S_0\,\Phi\,(\infty) - Ee^{-rT}\,\Phi\,(-\infty) = S_0.$$

6. If

$$V(0,S_0) = S_0^2\,e^{(\sigma^2+2r)T}\,\Phi\left(\frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right) - Ee^{-rT}\,\Phi\left(\frac{\log(S_0^2/E) + (2r-\sigma^2)T}{2\sigma\sqrt{T}}\right),$$

then

$$\begin{aligned}
\Delta = \frac{\partial V}{\partial S_0} &= 2S_0\,e^{(\sigma^2+r)T}\,\Phi\left(\frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right) \\
&\quad + S_0^2\,e^{(\sigma^2+2r)T}\,\Phi'\left(\frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right)\cdot\frac{1}{S_0\sigma\sqrt{T}} \\
&\quad - Ee^{-rT}\,\Phi'\left(\frac{\log(S_0^2/E) + (2r-\sigma^2)T}{2\sigma\sqrt{T}}\right)\cdot\frac{1}{S_0\sigma\sqrt{T}} \\
&= 2S_0\,e^{(\sigma^2+r)T}\,\Phi\left(\frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right) \\
&\quad + \frac{S_0}{\sigma\sqrt{T}}\,e^{(\sigma^2+r)T}\,\Phi'\left(\frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right) \\
&\quad - \frac{E}{S_0\sigma\sqrt{T}}e^{-rT}\,\Phi'\left(\frac{\log(S_0^2/E) + (2r-\sigma^2)T}{2\sigma\sqrt{T}}\right).
\end{aligned}$$

It turns out that this expression can be simplified. If we now write

$$d_1 = \frac{\log(S_0^2/E) + (2r+3\sigma^2)T}{2\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log(S_0^2/E) + (2r-\sigma^2)T}{2\sigma\sqrt{T}} = d_1 - 2\sigma\sqrt{T},$$

then

$$\Delta = 2S_0\,e^{(\sigma^2+r)T}\,\Phi\,(d_1) + \frac{1}{S_0\sigma\sqrt{T}}\left[S_0^2\,e^{(\sigma^2+r)T}\,\Phi'\,(d_1) - Ee^{-rT}\,\Phi'\,(d_2)\right].$$

Since

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

we conclude that

$$\begin{aligned}
\log\left[\frac{S_0^2\,e^{(\sigma^2+r)T}\,\Phi'\,(d_1)}{Ee^{-rT}\,\Phi'\,(d_2)}\right] &= \log(S_0^2/E) + \sigma^2T + 2rT - \frac{d_1^2}{2} + \frac{d_2^2}{2} \\
&= \log(S_0^2/E) + \sigma^2T + 2rT - \frac{1}{2}(d_2^2 - d_1^2).
\end{aligned}$$

Notice that

$$d_2^2 - d_1^2 = (d_2 - d_1)(d_2 + d_1).$$

Since

$$d_2 - d_1 = -2\sigma\sqrt{T}$$

and

$$d_2 + d_1 = \frac{2\log(S_0^2/E) + (4r + 2\sigma^2)T}{2\sigma\sqrt{T}}$$

we conclude that

$$d_2^2 - d_1^2 = -2\log(S_0^2/E) - 4rT - 2\sigma^2 T$$

and so

$$\log\left[\frac{S_0^2\, e^{(\sigma^2+r)T}\, \Phi'(d_1)}{Ee^{-rT}\, \Phi'(d_2)}\right] = \log(S_0^2/E) + \sigma^2 T + 2rT - \frac{1}{2}\left(2\log(S_0^2/E) - 4rT - 2\sigma^2 T\right) = 0.$$

This now implies that

$$S_0^2\, e^{(\sigma^2+r)T}\, \Phi'(d_1) - Ee^{-rT}\, \Phi'(d_2) = 0$$

so that

$$\Delta = 2S_0\, e^{(\sigma^2+r)T}\, \Phi\left(\frac{\log(S_0^2/E) + (2r + 3\sigma^2)T}{2\sigma\sqrt{T}}\right).$$