Statistics 441 Midterm – March 18, 2009

This exam has 6 problems on 6 numbered pages and is worth a total of 50 points.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

You must answer all of the questions in the space provided. Note that blank space is NOT an indication of a question's difficulty.

Name: ____________________________________

Instructor: Michael Kozdron

Problem	Score
1	
2	
3	
4	
5	
6	

TOTAL: ____________

1. (*9 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and let $\{\mathcal{F}_t, t \geq 0\}$ denote the Brownian filtration. Compute the following:

(a) $\mathbb{E}(B_4^2 + 3B_4 + 2)$,

(b) $\mathbb{E}(B_3^2 | \mathcal{F}_2)$, and

(c) $\lim_{n\to\infty} \sum_{j=1}^{2n} (B_{j/n} - B_{(j-1)/n})^2$.

2. (*8 points*)

(a) Determine the distribution of the Wiener integral $\int_0^2 \sqrt{s}\,\mathrm{d}B_s$.

(b) Determine the variance of the Itô integral $\int_0^1 e^{B_s}\,\mathrm{d}B_s$. *Hint: Using a moment generating function will help.*

3. (*12 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Use Itô's formula to determine the stochastic differential equations satisfied by $\{X_t, t \geq 0\}$ when

(a) $X_t = B_t^3$,

(b) $X_t = \sqrt{t}\, B_t$, and

(c) $X_t = Y_t^2$ where $\mathrm{d}Y_t = 2\sqrt{Y_t}\,\mathrm{d}B_t + 3\,\mathrm{d}t$.

4. (*8 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Let a, $b \in \mathbb{R}$ be real constants, and define the process $\{X_t, 0 \leq t < 1\}$ by setting

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{\mathrm{d}B_s}{1-s}.$$

Verify that X_t satisfies the stochastic differential equation

$$\mathrm{d}X_t = \frac{b - X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t, \quad 0 \leq t < 1.$$

The process $\{X_t, t \geq 0\}$ is known as a Brownian bridge from a to b.

5. (*5 points*) Suppose that a stock $\{S_t, t \geq 0\}$ follows geometric Brownian motion with volatility $\sigma > 0$ and drift μ, and assume that $r > 0$ denotes the risk-free interest rate. The explicit Black-Scholes solution $V(0, S_0)$ for the fair price (at time 0) of a European call option on this stock with payoff $\max\{S_T - E, 0\}$ at the expiry date T is given by

$$V(0, S_0) = S_0\, \Phi\left(\frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ee^{-rT}\, \Phi\left(\frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Determine the limit of $V(0, S_0)$ as the volatility σ approaches infinity, i.e., $\sigma \to \infty$.

6. (*8 points*) Suppose that a stock $\{S_t, t \geq 0\}$ follows geometric Brownian motion with volatility $\sigma > 0$ and drift μ, and assume that $r > 0$ denotes the risk-free interest rate. Consider the fair price (at time 0) of a European call option on this stock with a payoff of $\max\{S_T^2 - E, 0\}$ at the expiry date T. As we found in class, it is given by

$$V(0, S_0) = S_0^2 \, e^{(\sigma^2+r)T} \, \Phi\left(\frac{\log(S_0^2/E) + (2r + 3\sigma^2)T}{2\sigma\sqrt{T}}\right) - Ee^{-rT} \, \Phi\left(\frac{\log(S_0^2/E) + (2r - \sigma^2)T}{2\sigma\sqrt{T}}\right).$$

Compute delta (Δ) for this option value.