**Make sure that this examination has 10 numbered pages**

**University of Regina**
**Department of Mathematics & Statistics**
Final Examination
200910
(April 20, 2009)

**Statistics 441**
*Stochastic Calculus with Applications to Finance*

**Name:** ______________________ **Student Number:** ______________

**Instructor:** Michael Kozdron **Time:** 3 hours

**Read all of the following information before starting the exam.**

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.* ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.***

*You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.*

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

*Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.*

*This test has **10** numbered pages with **10** questions totalling **150** points. The number of points per question is indicated. For questions with multiple parts, all parts are equally weighted unless otherwise indicated.*

**Fact**: If $Z \sim \mathcal{N}(0, 1)$, then $\mathbf{P}\{-1.96 \le Z \le 1.96\} = 0.95$.

DO NOT WRITE BELOW THIS LINE

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|---|---|---|---|---|---|
| Problem 1 | ______ | Problem 2 | ______ | Problem 3 | ______ |
| Problem 4 | ______ | Problem 5 | ______ | Problem 6 | ______ |
| Problem 7 | ______ | Problem 8 | ______ | Problem 9 | ______ |
| Problem 10 | ______ | | | | |
| | | | | TOTAL | ______ |

**1.** (*16 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and let $\{\mathcal{F}_t, t \geq 0\}$ denote the Brownian filtration.

**(a)** (*5 pts*) Compute $\operatorname{Cov}(B_4 - B_2, B_3 - B_1)$.

**(b)** (*5 pts*) Compute $\mathbb{E}(B_4 - B_1 | \mathcal{F}_3)$.

**(c)** (*6 pts*) Now suppose that $\{W_t, t \geq 0\}$ is another standard Brownian motion starting at 0 that is independent of $\{B_t, t \geq 0\}$. Define the process $\{X_t, t \geq 0\}$ by setting

$$X_t = B_t W_t, \quad t \geq 0.$$

Show that $\{X_t, t \geq 0\}$ is *not* a standard Brownian motion starting at 0.

**2.** (*12 points*) Determine the distribution of the Riemann integral $\int_0^1 e^s B_s \, \mathrm{d}s$. *Hint: Use the integration by parts formula for Wiener integrals.*

**3.** (*18 points*) Suppose that $\{B_t, t \geq 0\}$ and $\{W_t, t \geq 0\}$ are independent standard Brownian motions with $B_0 = W_0 = 0$. Use an appropriate version of Itô's formula to determine $\mathrm{d}X_t$ when

**(a)** $X_t = t^2 B_t^2 + 1$,

**(b)** $X_t = e^{Y_t}$ where $\mathrm{d}Y_t = 2Y_t^2 \,\mathrm{d}B_t + 3Y_t \,\mathrm{d}t$, and

**(c)** $X_t = B_t^2 W_t^2 + tW_t$.

**4.** (*18 points*) Let $\{B_t, t \geq 0\}$ denote a standard Brownian motion with $B_0 = 0$. Determine whether or not each of the following processes is a martingale with respect to the Brownian filtration. Be sure to give reasons.

**(a)** $\{X_t, t \geq 0\}$ where $X_t = B_t^3$

**(b)** $\{Y_t, t \geq 0\}$ where $Y_t = \int_0^t B_s^2 \, \mathrm{d}B_s$

**(c)** $\{Z_t, t \geq 0\}$ where $Z_t = B_t^3 - B_t^2 - 3tB_t + t$

**5.** (*12 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Let $\{X_t, t \geq 0\}$ denote geometric Brownian motion given by

$$X_t = X_0 \exp\{\sigma B_t + \mu t\}$$

where $X_0 > 0$, $\sigma > 0$, and $\mu \in \mathbb{R}$ are given. If $T > 0$, determine the values of $L$ and $U$ such that

$$\mathbf{P}\{L \leq X_T \leq U\} = 0.95.$$

Note that $L$ and $U$ will necessarily depend on $X_0$, $\sigma$, $\mu$, and $T$. Of course, we call $[L, U]$ a 95% confidence interval for $X_T$.

**6.** (*14 points*) Suppose that the financial position $X$ is known to have an exponential distribution with mean $1/2$ so that the density of $X$ is

$$f_X(x) = 2e^{-2x}, \quad x > 0.$$

**(a)** Determine $\mathrm{VaR}_\alpha(X)$ for $0 < \alpha < 1$.

**(b)** Recall that for $0 < \alpha < 1$, the average value at risk at level $\alpha$ is given by

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_x(X)\,\mathrm{d}x.$$

Compute $\mathrm{AVaR}_\alpha(X)$ if $X$ has an exponential distribution with mean $1/2$. *Hint: Use integration by parts to evaluate the resulting integral explicitly.*

**7.** (*10 points*) Let $\Omega$ denote the space of all possible financial scenarios of interest, and let $\mathcal{X}$ denote the space of functions $X : \Omega \to \mathbb{R}$ with

$$\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)| < \infty.$$

Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a monetary risk measure. Prove that if $\rho$ is subadditve and positively homogeneous, then $\rho$ is convex.

**8.** (*18 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Consider the process $\{X_t, t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}X_t = a(t)X_t\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t, \quad X_0 = x,$$

where $a$ is a continuous function of one variable and $b$ is a continuous function of two variables.

**(a)** (*14 pts*) Define the *integrating factor*

$$Y_t = \exp\left\{-\int_0^t a(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t a^2(s)\,\mathrm{d}s\right\}.$$

Use Itô's formula and the chain rule, as appropriate, to show that $\mathrm{d}(X_tY_t) = Y_t\,b(t, X_t)\,\mathrm{d}t$.

**(b)** (*4 pts*) Now define $Z_t = X_tY_t$ so that $X_t = Y_t^{-1}Z_t$. Deduce immediately from **(a)** that

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = Y_t\,b(t, Y_t^{-1}Z_t), \quad Z_0 = x.$$

*This is now a deterministic ordinary differential equation that can be solved using first-year calculus!*

**9.** (*14 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Consider the process $\{X_t, t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}X_t = \alpha X_t \,\mathrm{d}B_t + \frac{1}{X_t}\,\mathrm{d}t, \quad X_0 = x > 0,$$

where $\alpha$ is a constant. Use the method outlined in the previous problem to show that the solution to this SDE is

$$X_t = \exp\left\{\alpha B_t - \frac{1}{2}\alpha^2 t\right\}\left[x^2 + 2\int_0^t \exp\{-2\alpha B_s + \alpha^2 s\}\,\mathrm{d}s\right]^{1/2}.$$

*Hint: Derive the ordinary differential equation as in (b) of Problem 8, separate variables, and integrate both sides. Since $Z$ is deterministic, the integral with respect to $Z$ is just an ordinary Riemann integral.*

*Note that the explicit solution to the SDE shows that the process will remain positive for all time. Thus, it can be used as a model for a stock price within the Black-Scholes framework.*

**10.** (*18 points*) The purpose of this problem is to lead you through the solution of the pricing problem for a *digital call option* (sometimes known as a *cash-or-nothing call option* or a *binary call option*) in the Black-Scholes framework.

Suppose that our stock price of interest $\{S_t, t \geq 0\}$ follows geometric Brownian motion with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$ so that it satisfies the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}\tilde{B}_t + \mu S_t \,\mathrm{d}t, \quad S_0 > 0,$$

where $\{\tilde{B}_t, t \geq 0\}$ is a standard Brownian motion starting at 0.

**(a)** (*4 pts*) Assuming that the risk-free interest rate is $r > 0$, write down the stochastic differential equation satisfied by the associated risk-neutral process $\{X_t, t \geq 0\}$.

**(b)** (*4 pts*) If $T > 0$ is the expiry date, the payoff function for a digital call option with strike price $E$ is

$$\Lambda(x) = \begin{cases} 1, & \text{if } x \geq E, \\ 0, & \text{if } x < E. \end{cases}$$

Write down an expression for $V(0, S_0)$, the fair price to pay for this option at time 0, which involves an expectation of $\Lambda(X_T)$.

**(c)** (*10 pts*) Evaluate the expression from **(b)** for $V(0, S_0)$. *Your answer will involve* $\Phi$*, the standard normal cumulative distribution function.*