Stat 441 Winter 2009
Assignment #7

This assignment is due at the beginning of class on Friday, March 6, 2009. Note that the Midterm on Wednesday, March 18, 2009, will cover material through Lecture #22 including this assignment.

1. Read Chapter 8, pages 73–86, and Chapter 10, pages 99–104, of Higham.

- Complete Exercise 10.6 on page 102.
- Experiment with the MATLAB program in Chapter 10 (which is an extended version of the program from Chapter 8. It outputs the value of a call, the delta of a call, the vega of a call, the value of a put, the delta of a put, and the vega of a put. Note that Higham gives the general Black-Scholes formula for any time $0 \le t \le T$, and he writes $\tau = T - t$. Since we are considering the option value at time $t = 0$, we are taking $\tau = T - 0 = T$.
- Complete Exercise P10.1 on page 104.

2. If a stock moving according to geometric Brownian motion has a constant volatility of 18% and constant drift of 8%, with continuously compounded interest rate constant at 6%, what is the value of an option to buy the stock for \$25 in two years time, given a current stock price of \$20?

Verify the following. The description fits the Black-Scholes conditions so that using $S_0 = 20$, $E = 25$, $\sigma = 0.18$, $r = 0.06$, $t = 0$, and $T = 2$, we find $V(0, S_0) = \$1.221$.

3. A stock has current price \$10 and moves as a geometric Brownian motion with upward drift of 15% a year and with volatility of 20% a year. Current interest rates are constant at 5%.

(a) What is the value of a call option for \$12 on the stock in one year's time? Answer this question by working out the Black-Scholes formula by hand.

(b) Now determine the delta, gamma, rho, theta, and vega of the call option described in (a). Answer this question by working out the Greeks by hand.

(c) Verify your answers to (a) and (b) by using the Higham's MATLAB program for Chapter 10.

(d) Write a spreadsheet that computes the Black-Scholes option value. To paraphrase from `http://www.espenhaug.com/black_scholes.html`: "Are you too lazy to type in what you see above? Okay download me here."

4. Read Chapter 14, pages 131–139, of Higham.

- Experiment with the MATLAB program in Chapter 14.
- Complete Exercise P14.2 on page 138.

5. The purpose of this problem is to outline the solution to the stochastic differential equation $\mathrm{d}X_t = \sigma(t) X_t \, \mathrm{d}B_t$ where $\sigma(t)$ is a deterministic functions of time and $\{B_t, t \ge 0\}$ is a standard Brownian motion.

(continued)

Begin by recalling that the solution to

$$\mathrm{d}X_t = \sigma X_t \,\mathrm{d}B_t \tag{$*$}$$

where σ is a constant is given by

$$X_t = X_0 \exp\left\{\sigma B_t - \frac{\sigma^2}{2} t\right\}.$$

This can be checked easily with Itô's formula. One way to derive the solution, however, is to consider the equation written in the form

$$\frac{\mathrm{d}X_t}{X_t} = \sigma \,\mathrm{d}B_t.$$

The left side now looks like the derivative of the logarithm function. In fact, if X_t is deterministic, then

$$\mathrm{d}\log(X_t) = \frac{\mathrm{d}X_t}{X_t}.$$

However, since X_t is random, we need to use Itô's formula; that is,

$$\mathrm{d}\log(X_t) = \frac{\mathrm{d}X_t}{X_t} - \frac{\mathrm{d}\langle X\rangle_t}{2X_t^2}. \tag{$**$}$$

From $(*)$ we see that

$$\mathrm{d}\langle X\rangle_t = \sigma^2 X_t^2 \,\mathrm{d}t.$$

Now we can substitute into $(**)$ and conclude

$$\mathrm{d}\log(X_t) = \frac{\mathrm{d}X_t}{X_t} - \frac{\mathrm{d}\langle X\rangle_t}{2X_t^2} = \frac{\sigma X_t \,\mathrm{d}B_t}{X_t} - \frac{\sigma^2 X_t^2 \,\mathrm{d}t}{2X_t^2} = \sigma \,\mathrm{d}B_t - \frac{\sigma^2}{2}\,\mathrm{d}t.$$

Since we have succeeded in removing X_t from the right side of the equation, we can simply integrate both sides from 0 to T. Thus,

$$\int_0^T \mathrm{d}\log(X_t) = \int_0^T \left[\sigma \,\mathrm{d}B_t - \frac{\sigma^2}{2}\,\mathrm{d}t\right].$$

As for the left side, we find

$$\int_0^T \mathrm{d}\log(X_t) = \log(X_T) - \log(X_0) = \log\left(\frac{X_T}{X_0}\right)$$

(don't forget you need both limits of integration here) and for the right side

$$\int_0^T \left[\sigma \,\mathrm{d}B_t - \frac{\sigma^2}{2}\,\mathrm{d}t\right] = \sigma \int_0^T \mathrm{d}B_t - \frac{\sigma^2}{2}\int_0^T \mathrm{d}t = \sigma(B_T - B_0) - \frac{\sigma^2}{2}(T-0) = \sigma B_T - \frac{\sigma^2}{2}T.$$

That is,

$$\log\left(\frac{X_T}{X_0}\right) = \sigma B_T - \frac{\sigma^2}{2}T$$

and so solving for X_T gives

$$X_T = X_0 \exp\left\{\sigma B_T - \frac{\sigma^2}{2}T\right\}.$$

(We can now switch the dummy variable T to t to match our earlier result.)

(continued)

You can now use exactly the same technique to solve

$$dX_t = \sigma(t) X_t \, dB_t$$

where $\sigma(t)$ is a deterministic functions of time. (Note that the solution involves the exponential of a Wiener integral, however.) Actually, this exact same technique works to solve

$$dX_t = \sigma(t) X_t \, dB_t + \mu(t) \, dt$$

where both $\sigma(t)$ and $\mu(t)$ are deterministic functions of time.

6. The following exercises are from the printed lecture notes.

- Exercises 19.1, 19.2, and 19.3.
- Exercise 20.1.
- Exercises 21.1, 21.2, 21.3, 21.5