Stat 441 Winter 2009
Assignment #5

This assignment is due at the beginning of class on Friday, February 13, 2009.

1. The following exercises are from the printed lecture notes.

- Exercise 13.4
- Exercises 14.7 and 14.8
- Exercise 16.6

2. Suppose that $\{B_t, t \geq 0\}$ is a Brownian motion starting at 0. If the process $\{X_t, t \geq 0\}$ is defined by setting

$$X_t = \exp\{B_t\},$$

use Itô's formula to compute $\mathrm{d}X_t$.

3. Suppose that the price of a stock $\{X_t, t \geq 0\}$ follows geometric Brownian motion with drift 0.05 and volatility 0.3 so that it satisfies the stochastic differential equation

$$\mathrm{d}X_t = 0.3 X_t \, \mathrm{d}B_t + 0.05 X_t \, \mathrm{d}t.$$

If the price of the stock at time 2 is 30, determine the probability that the price of the stock at time 2.5 is between 30 and 33.

4. Consider the Itô process $\{X_t, t \geq 0\}$ described by the stochastic differential equation

$$\mathrm{d}X_t = 0.10 X_t \, \mathrm{d}B_t + 0.25 X_t \, \mathrm{d}t.$$

Calculate the probability that X_t is at least 5% higher than X_0

(a) at time $t = 0.01$, and

(b) at time $t = 1$.

5. Consider the Itô process $\{X_t, t \geq 0\}$ described by the stochastic differential equation

$$\mathrm{d}X_t = 0.05 X_t \, \mathrm{d}B_t + 0.1 X_t \, \mathrm{d}t, \quad X_0 = 35.$$

Compute $\mathbf{P}\{X_5 \leq 48\}$.

6. Consider the Itô process $\{Y_t, t \geq 0\}$ described by the stochastic differential equation

$$\mathrm{d}Y_t = 0.4 \, \mathrm{d}B_t + 0.1 \, \mathrm{d}t.$$

If the process $\{X_t, t \geq 0\}$ is defined by $X_t = e^{0.5 Y_t}$, determine $\mathrm{d}X_t$.

7. Suppose that the process $\{X_t, t \geq 0\}$ is defined by the stochastic differential equation

$$\mathrm{d}X_t = \sigma X_t \,\mathrm{d}B_t + \mu(t) X_t \,\mathrm{d}t$$

where the volatility σ is constant, but the drift $\mu(t)$ is a function of time. Determine (an expression for) X_t (assuming sufficient regularity of the function μ).

8. Suppose that $g : \mathbb{R} \to [0, \infty)$ is a bounded, piecewise continuous, deterministic function. Assume further that $g \in L^2([0,\infty))$ so that the Wiener integral

$$I_t = \int_0^t g(s) \,\mathrm{d}B_s$$

is well defined for all $t \geq 0$. Define the continuous-time stochastic process $\{M_t, t \geq 0\}$ by setting

$$M_t = I_t^2 - \int_0^t g^2(s) \,\mathrm{d}s = \left(\int_0^t g(s) \,\mathrm{d}B_s \right)^2 - \int_0^t g^2(s) \,\mathrm{d}s.$$

Use Itô's formula to prove that $\{M_t, t \geq 0\}$ is a continuous-time martingale.

9. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Consider the process $\{Y_t, t \geq 0\}$ defined by setting $Y_t = B_t^k$ where k is a positive integer. Use Itô's formula to show that Y_t satisfies the SDE

$$\mathrm{d}Y_t = k Y_t^{1-1/k} \,\mathrm{d}B_t + \frac{k(k-1)}{2} Y_t^{1-2/k} \,\mathrm{d}t.$$

10. Consider the following time-inhomogeneous Ornstein-Uhlenbeck-type process

$$\mathrm{d}X_t = \sigma(t) \,\mathrm{d}B_t - a(X_t - g(t)) \,\mathrm{d}t$$

where g and σ are (sufficiently regular) deterministic functions of time.

(a) Find an explicit expression for the solution X_t of the above SDE (in terms of integrals involving g and σ with respect to B_t).

(b) Let $Y_t = \exp\{X_t + ct\}$. Use Itô's formula to compute $\mathrm{d}Y_t$.