Lecture #9, 10: Riemann Integration

Suppose that $g : [a, b] \to \mathbb{R}$ is a real-valued function on $[a, b]$. Fix a positive integer n, and let

$$\pi_n = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$$

be a *partition* of $[a, b]$. For $i = 1, \cdots, n$, define $\Delta t_i = t_i - t_{i-1}$ and let $t_i^* \in [t_{i-1}, t_i]$ be distinguished points; write $\tau_n^* = \{t_1^*, \ldots, t_n^*\}$ for the set of distinguished points. If π_n is a partition of $[a, b]$, define the *mesh* of π_n to be the width of the largest subinterval; that is,

$$\text{mesh}(\pi_n) = \max_{1 \le i \le n} \Delta t_i = \max_{1 \le i \le n} (t_i - t_{i-1}).$$

Finally, we call

$$S(g; \pi_n; \tau_n^*) = \sum_{i=1}^n g(t_i^*) \Delta t_i$$

the *Riemann sum for g corresponding to the partition π_n with distinguished points τ_n^*.*

We say that $\pi = \{\pi_n, n = 1, 2, \ldots\}$ is a *refinement* of $[a, b]$ if π is a sequence of partitions of $[a, b]$ with $\pi_n \subset \pi_{n+1}$ for all n.

Definition 6.1. We say that g is *Riemann integrable over* $[a, b]$ and define the *Riemann integral* of g to be I if for every $\varepsilon > 0$ and for every refinement $\pi = \{\pi_n, n = 1, 2, \ldots\}$ with $\text{mesh}(\pi_n) \to 0$ as $n \to \infty$, there exists an N such that

$$|S(g; \pi_m; \tau_m^*) - I| < \varepsilon$$

for all choices of distinguished points τ_m^* and for all $m \ge N$. We then define

$$\int_a^b g(s) \, \mathrm{d}s$$

to be this limiting value I.

Remark. There are various equivalent definitions of the Riemann integral including Darboux's version using upper and lower sums. The variant given in Definition 6.1 above will be the most useful one for our construction of the stochastic integral.

The following theorem gives a sufficient condition for a function to be Riemann integrable.

Theorem 6.2. *If $g : [a, b] \to \mathbb{R}$ is bounded and piecewise continuous, then g is Riemann integrable on $[a, b]$.*

Proof. For a proof, see Theorem 6.10 on page 126 of Rudin [22]. □

The previous theorem is adequate for our purposes. However, it is worth noting that, in fact, this theorem follows from a more general result which completely characterizes the class of Riemann integrable functions.

Theorem 6.3. *Suppose that $g : [a, b] \to \mathbb{R}$ is bounded. The function g is Riemann integrable on $[a, b]$ if and only if the set of discontinuities of g has Lebesgue measure 0.*

Proof. For a proof, see Theorem 11.33 on page 323 of Rudin [22]. $\square$

There are two particular Riemann sums that are studied in elementary calculus—the so-called *left-hand Riemann sum* and *right-hand Riemann sum.*

For $i = 0, 1, \dots, n$, let $t_i = a + \frac{i(b-a)}{n}$. If $t_i^* = t_{i-1}$, then

$$\frac{b-a}{n} \sum_{i=1}^{n} g\left(a + \frac{(i-1)(b-a)}{n}\right)$$

is called the *left-hand Riemann sum.* The *right-hand Riemann sum* is obtained by choosing $t_i^* = t_i$ and is given by

$$\frac{b-a}{n} \sum_{i=1}^{n} g\left(a + \frac{i(b-a)}{n}\right).$$

Remark. It is a technical matter that if $t_i = a + \frac{i(b-a)}{n}$, then $\pi = \{\pi_n, n = 1, 2, \ldots\}$ with $\pi_n = \{t_0 = a < t_1 < \cdots < t_{n-1} < t_n = b\}$ is not a refinement. To correct this, we simply restrict to those n of the form $n = 2^k$ for some k in order to have a refinement of $[a, b]$. Hence, from now on, we will not let this concern us.

The following example shows that even though the limits of the left-hand Riemann sums and the right-hand Riemann sums might both exist and be equal for a function g, that is not enough to guarantee that g is Riemann integrable.

Example 6.4. Suppose that $g : [0, 1] \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ 1, & \text{if } x \notin \mathbb{Q} \cap [0, 1]. \end{cases}$$

Let $\pi_n = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\}$ so that $\Delta t_i = \frac{1}{n}$ and $\operatorname{mesh}(\pi_n) = \frac{1}{n}$. The limit of the left-hand Riemann sums is therefore given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g\left(\frac{i-1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(0) = 0$$

since $\frac{i-1}{n}$ is necessarily rational. Similarly, the limit of the right-hand Riemann sums is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g\left(\frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(0) = 0.$$

However, define a sequence of partitions as follows:

$$\pi_n = \left\{0 < \frac{1}{n\sqrt{2}} < \cdots < \frac{n-1}{n\sqrt{2}} < \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} + \frac{\sqrt{2}-1}{n\sqrt{2}} < \cdots < \frac{1}{\sqrt{2}} + \frac{(n-1)(\sqrt{2}-1)}{n\sqrt{2}} < 1\right\}.$$

In this case, $\text{mesh}(\pi_n) = \frac{\sqrt{2}-1}{n\sqrt{2}}$ so that $\text{mesh}(\pi_n) \to 0$ as $n \to \infty$. If t_i^* is chosen to be the mid-point of each interval, then t_i^* is necessarily irrational so that $g(t_i^*) = 1$. Therefore,

$$\sum_{i=1}^{n} g(t_i^*)\Delta t_i = \sum_{i=1}^{n} \Delta t_i = \sum_{i=1}^{n}(t_i - t_{i-1}) = t_n - t_0 = 1 - 0 = 1$$

for each n. Hence, we conclude that g is *not* Riemann integrable on $[0, 1]$ since there is no unique limiting value.

However, we can make the following postive assertion about the limits of the left-hand Riemann sums and the right-hand Riemann sums.

Remark. Suppose that $g : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ so that

$$I = \int_a^b g(s)\,\mathrm{d}s$$

exists. Then, the limit of the left-hand Riemann sums and the limit of the right-hand Riemann sums both exist, and furthermore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g\left(\frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g\left(\frac{i-1}{n}\right) = I.$$