Lecture #35: The Characteristic Function for Heston's Model

As we saw last lecture, it is sometimes possible to determine the characteristic function of a random variable defined via a stochastic differential equation without actually solving the SDE. The computation involves the Feynman-Kac representation theorem, but it does require the solution of a partial differential equation. In certain cases where an explicit solution does not exist for the SDE, computing the characteristic function might still be possible as long as the resulting PDE is solvable.

Recall that the Heston model assumes that the asset price process $\{S_t, t \ge 0\}$ satisfies the SDE

$$\mathrm{d}S_t = \sqrt{v_t}\, S_t \,\mathrm{d}B_t^{(1)} + \mu S_t \,\mathrm{d}t$$

where the variance process $\{v_t, t \ge 0\}$ satisfies

$$\mathrm{d}v_t = \sigma\sqrt{v_t}\,\mathrm{d}B_t^{(2)} + a(b - v_t)\,\mathrm{d}t$$

and the two driving Brownian motions $\{B_t^{(1)}, t \ge 0\}$ and $\{B_t^{(2)}, t \ge 0\}$ are correlated with rate ρ, i.e.,

$$\mathrm{d}\langle B^{(1)}, B^{(2)}\rangle_t = \rho\,\mathrm{d}t.$$

In order to analyze the Heston model, it is easier to work with

$$X_t = \log(S_t)$$

instead. Itô's formula implies that $\{X_t, t \ge 0\}$ satisfies the SDE

$$\mathrm{d}X_t = \mathrm{d}\log S_t = \frac{\mathrm{d}S_t}{S_t} - \frac{\mathrm{d}\langle S\rangle_t}{2S_t^2} = \sqrt{v_t}\;\mathrm{d}B_t^{(1)} + \left(\mu - \frac{v_t}{2}\right)\,\mathrm{d}t.$$

We will now determine the characteristic function of X_T for any $T \ge 0$. The multidimensional version of Itô's formula (Theorem 20.4) implies that

$$\begin{aligned}
\mathrm{d}f(t, X_t, v_t) &= \dot{f}(t, X_t, v_t)\,\mathrm{d}t + f_1(t, X_t, v_t)\,\mathrm{d}X_t + \frac{1}{2}f_{11}(t, X_t, v_t)\,\mathrm{d}\langle X\rangle_t \\
&\qquad + f_2(t, X_t, v_t)\,\mathrm{d}v_t + \frac{1}{2}f_{22}(t, X_t, v_t)\,\mathrm{d}\langle v\rangle_t + f_{12}(t, X_t, v_t)\,\mathrm{d}\langle X, v\rangle_t \\
&= \dot{f}(t, X_t, v_t)\,\mathrm{d}t + f_1(t, X_t, v_t)\left(\sqrt{v_t}\;\mathrm{d}B_t^{(1)} + \left(\mu - \frac{v_t}{2}\right)\,\mathrm{d}t\right) + \frac{1}{2}f_{11}(t, X_t, v_t)v_t\,\mathrm{d}t \\
&\qquad + f_2(t, X_t, v_t)\big(\sigma\sqrt{v_t}\,\mathrm{d}B_t^{(2)} + a(b - v_t)\,\mathrm{d}t\big) + \frac{1}{2}f_{22}(t, X_t, v_t)\sigma^2 v_t\,\mathrm{d}t \\
&\qquad\qquad + f_{12}(t, X_t, v_t)\sigma\rho v_t\,\mathrm{d}t \\
&= f_1(t, X_t, v_t)\sqrt{v_t}\;\mathrm{d}B_t^{(1)} + f_2(t, X_t, v_t)\sigma\sqrt{v_t}\,\mathrm{d}B_t^{(2)} + (\mathcal{A}f)(t, X_t, v_t)\,\mathrm{d}t
\end{aligned}$$

where the differential operator $\mathcal{A}$ is defined as

$$(\mathcal{A}f)(t,x,y) = \dot{f}(t,x,y) + \left(\mu - \frac{y}{2}\right) f_1(t,x,y) + \frac{y}{2} f_{11}(t,x,y) + a(b-y) f_2(t,x,y) + \frac{\sigma^2 y}{2} f_{22}(t,x,y) + \sigma\rho y f_{12}(t,x,y).$$

If we now let $u(x) = e^{i\theta x}$, then the (multidimensional form of the) Feynman-Kac representation theorem implies

$$f(t,x,y) = \mathbb{E}[u(X_T)|X_t = x, v_t = y] = \mathbb{E}[e^{i\theta X_T}|X_t = x, v_t = y]$$

is the unique bounded solution of the partial differential equation

$$(\mathcal{A}f)(t,x,y) = 0, \quad 0 \le t \le T, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \tag{22.1}$$

subject to the terminal condition

$$f(T,x,y) = e^{i\theta x}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

Note that $f(0,x,y) = \mathbb{E}[e^{i\theta X_T}|X_0 = x, v_0 = y] = \varphi_{X_T}(\theta)$ is the characteristic function of X_T.

Guided by the form of the terminal condition and by our experience with the Ornstein-Uhlenbeck characteristic function, we guess that $f(t,x,y)$ can be written as

$$f(t,x,y) = \exp\{\alpha(t) y + \beta(t)\} \exp\{i\theta x\} \tag{22.2}$$

for some functions $\alpha(t)$ and $\beta(t)$ of t only satisfying $\alpha(T) = 0$ and $\beta(T) = 0$. Differentiating we find

$$\dot{f}(t,x,y) = [\alpha'(t) y + \beta'(t)] f(t,x,y), \quad f_1(t,x,y) = i\theta f(t,x,y), \quad f_{11}(t,x,y) = -\theta^2 f(t,x,y),$$

$$f_2(t,x,y) = \alpha(t) f(t,x,y), \quad f_{22}(t,x,y) = \alpha^2(t) f(t,x,y), \quad f_{12}(t,x,y) = i\theta\alpha(t) f(t,x,y),$$

so that substituting into the explicit form of $(\mathcal{A}f)(t,x,y) = 0$ and factoring out the common $f(t,x,y)$ gives

$$[\alpha'(t) y + \beta'(t)] + i\theta\left(\mu - \frac{y}{2}\right) - \frac{\theta^2}{2} y + a\alpha(t)(b-y) + \frac{\sigma^2\alpha^2(t)}{2} y + i\sigma\rho\theta\alpha(t) y = 0,$$

or equivalently,

$$\left[\alpha'(t) + (i\sigma\rho\theta - a)\alpha(t) + \frac{\sigma^2\alpha^2(t)}{2} - \frac{i\theta}{2} - \frac{\theta^2}{2}\right] y + \beta'(t) + i\theta\mu + ab\alpha(t) = 0.$$

Since this equation must be true for all $0 \le t \le T$, $x \in \mathbb{R}$, and $y \in \mathbb{R}$, the only way that is possible is if the coefficient of y is zero *and* the constant term is 0. Thus, we must have

$$\alpha'(t) + (i\sigma\rho\theta - a)\alpha(t) + \frac{\sigma^2\alpha^2(t)}{2} - \frac{i\theta}{2} - \frac{\theta^2}{2} = 0 \quad \text{and} \quad \beta'(t) + i\theta\mu + ab\alpha(t) = 0. \tag{22.3}$$

The first equation in (22.3) involves $\alpha(t)$ only and is of the form

$$\alpha'(t) = A\alpha(t) + B\alpha^2(t) + C$$

with

$$A = a - i\sigma\rho\theta, \quad B = -\frac{\sigma^2}{2}, \quad C = \frac{i\theta}{2} + \frac{\theta^2}{2}. \tag{22.4}$$

This ordinary differential equation can be solved by integration; see Exercise 22.1 below. The solution is given by

$$\alpha(t) = D + E\tan(Ft + G)$$

where

$$D = -\frac{A}{2B}, \quad E = \sqrt{\frac{C}{B} - \frac{A^2}{4B^2}}, \quad F = BE = B\sqrt{\frac{C}{B} - \frac{A^2}{4B^2}}, \tag{22.5}$$

and G is an arbitrary constant. The terminal condition $\alpha(T) = 0$ implies

$$0 = D + E\tan(FT + G) \quad \text{so that} \quad G = \arctan\left(-\frac{D}{E}\right) - FT$$

which gives

$$\alpha(t) = D + E\tan\left(\arctan\left(-\frac{D}{E}\right) - F(T-t)\right). \tag{22.6}$$

Exercise 22.1. Suppose that a, b, and c are non-zero real constants. Compute

$$\int \frac{\mathrm{d}x}{ax^2 + bx + c}.$$

Hint: Complete the square in the denominator. The resulting function is an antiderivative of an arctangent function.

In order to simplify the expression for $\alpha(t)$ given by (22.6) above, we begin by noting that

$$\cos\left(\arctan\left(-\frac{D}{E}\right)\right) = \frac{E}{\sqrt{D^2+E^2}} \quad \text{and} \quad \sin\left(\arctan\left(-\frac{D}{E}\right)\right) = -\frac{D}{\sqrt{D^2+E^2}}. \tag{22.7}$$

Using the sum of angles identity for cosine therefore gives

$$\begin{aligned}
&\cos\left(\arctan\left(-\frac{D}{E}\right) - F(T-t)\right) \\
&\quad = \cos\left(\arctan\left(-\frac{D}{E}\right)\right)\cos\left(F(T-t)\right) + \sin\left(\arctan\left(-\frac{D}{E}\right)\right)\sin\left(F(T-t)\right) \\
&\quad = \frac{E}{\sqrt{D^2+E^2}}\cos\left(F(T-t)\right) - \frac{D}{\sqrt{D^2+E^2}}\sin\left(F(T-t)\right) \\
&\quad = \frac{E\cos\left(F(T-t)\right) - D\sin\left(F(T-t)\right)}{\sqrt{D^2+E^2}}.
\end{aligned} \tag{22.8}$$

Similarly, the sum of angles identity for sine yields

$$\sin\left(\arctan\left(-\frac{D}{E}\right) - F(T-t)\right) = \frac{-D\cos\left(F(T-t)\right) - E\sin\left(F(T-t)\right)}{\sqrt{D^2+E^2}}. \tag{22.9}$$

Writing $\tan(z) = \frac{\sin(z)}{\cos(z)}$ and using (22.8) and (22.9) implies

$$\begin{aligned}\tan\left(\arctan\left(-\frac{D}{E}\right) - F(T-t)\right) &= \frac{-D\cos\left(F(T-t)\right) - E\sin\left(F(T-t)\right)}{E\cos\left(F(T-t)\right) - D\sin\left(F(T-t)\right)} \\ &= \frac{-D\cot\left(F(T-t)\right) - E}{E\cot\left(F(T-t)\right) - D}\end{aligned}$$

so that substituting the above expression into (22.6) for $\alpha(t)$ gives

$$\alpha(t) = D + E\left[\frac{-D\cot\left(F(T-t)\right) - E}{E\cot\left(F(T-t)\right) - D}\right] = \frac{-(D^2+E^2)}{E\cot\left(F(T-t)\right) - D}.$$

The next step is to substitute back for D, E, and F in terms of the original parameters. It turns out, however, that it is useful to write them in terms of

$$\gamma = \sqrt{\sigma^2(\theta^2 + i\theta) + (a - i\sigma\rho\theta)^2}. \tag{22.10}$$

Thus, substituting (22.4) into (22.5) gives

$$D = \frac{a - i\sigma\rho\theta}{\sigma^2}, \quad E = \frac{i\gamma}{\sigma^2}, \quad \text{and} \quad F = -\frac{i\gamma}{2}. \tag{22.11}$$

Since

$$D^2 + E^2 = -\frac{i\theta + \theta^2}{\sigma^2}$$

we conclude that

$$\alpha(t) = \frac{i\theta + \theta^2}{i\gamma\cot\left(-\frac{i\gamma(T-t)}{2}\right) - (a - i\sigma\rho\theta)}.$$

The final simplification is to note that

$$\cos(-iz) = \cosh(z) \quad \text{and} \quad \sin(-iz) = -i\sinh(z)$$

so that

$$\cot(iz) = \frac{\cos(iz)}{\sin(iz)} = \frac{\cosh(z)}{-i\sinh(z)} = i\coth(z)$$

which gives

$$\alpha(t) = \frac{i\theta + \theta^2}{i^2\gamma\coth\left(\frac{\gamma(T-t)}{2}\right) - (a - i\sigma\rho\theta)} = -\frac{i\theta + \theta^2}{\gamma\coth\left(\frac{\gamma(T-t)}{2}\right) + (a - i\sigma\rho\theta)}.$$

Finally, we find

$$\exp\{\alpha(t)y\} = \exp\left\{-\frac{(i\theta + \theta^2)y}{\gamma\coth\left(\frac{\gamma(T-t)}{2}\right) + (a - i\sigma\rho\theta)}\right\}. \tag{22.12}$$

Having determined $\alpha(t)$, we can now consider the second equation in (22.3) involving $\beta'(t)$. It is easier, however, to manipulate this expression using $\alpha(t)$ in the form (22.6). Thus, the expression for $\beta'(t)$ now becomes

$$\beta'(t) = -abD - i\theta\mu - abE\tan\left(\arctan\left(-\frac{D}{E}\right) - F(T-t)\right)$$

which can be solved by integrating from 0 to t. Recall that

$$\int \tan(z)\,\mathrm{d}z = \log(\sec(z)) = -\log(\cos(z))$$

and so

$$\begin{aligned}\beta(t) &= \beta(0) - abDt - i\theta\mu t - abE\int_0^t \tan\left(\arctan\left(-\frac{D}{E}\right) - F(T-s)\right)\,\mathrm{d}s \\ &= \beta(0) - abDt - i\theta\mu t - \frac{abE}{F}\log\left(\frac{\cos(\arctan\left(-\frac{D}{E}\right) - FT)}{\cos(\arctan\left(-\frac{D}{E}\right) - F(T-t))}\right).\end{aligned}$$

The terminal condition $\beta(T) = 0$ implies that

$$\beta(0) = abDT + i\theta\mu T + \frac{abE}{F}\log\left(\frac{\sqrt{E^2+D^2}\cos(\arctan\left(-\frac{D}{E}\right) - FT)}{E}\right)$$

using (22.7), and so we now have

$$\beta(t) = abD(T-t) + i\theta\mu(T-t) + \frac{abE}{F}\log\left(\frac{\sqrt{E^2+D^2}\cos(\arctan\left(-\frac{D}{E}\right) - F(T-t))}{E}\right).$$

As in the calculation of $\alpha(t)$, we can simplify this further using (22.8) so that

$$\beta(t) = abD(T-t) + i\theta\mu(T-t) + \frac{abE}{F}\log\left(\cos\left(F(T-t)\right) - \frac{D}{E}\sin\left(F(T-t)\right)\right)$$

which implies

$$\exp\{\beta(t)\} = \exp\{abD(T-t)+i\theta\mu(T-t)\}\left(\cos\left(F(T-t)\right) - \frac{D}{E}\sin\left(F(T-t)\right)\right)^{\frac{abE}{F}}. \quad (22.13)$$

Substituting the expressions given by (22.11) for D, E, and F in terms of the original parameters into (22.13) gives

$$\exp\{\beta(t)\} = \frac{\exp\left\{\frac{ab(a-i\sigma\rho\theta)(T-t)}{\sigma^2} + i\theta\mu(T-t)\right\}}{\left(\cos\left(-\frac{i\gamma}{2}(T-t)\right) - \frac{a-i\sigma\rho\theta}{i\gamma}\sin\left(-\frac{i\gamma}{2}(T-t)\right)\right)^{\frac{2ab}{\sigma^2}}}.$$

As in the calculation of $\alpha(t)$, the final simplification is to note that $\cos(-iz) = \cosh(z)$ and $\sin(-iz) = -i\sinh(z)$ so that

$$\exp\{\beta(t)\} = \frac{\exp\left\{\frac{ab(a-i\sigma\rho\theta)(T-t)}{\sigma^2} + i\theta\mu(T-t)\right\}}{\left(\cosh\left(\frac{\gamma(T-t)}{2}\right) + \frac{a-i\sigma\rho\theta}{\gamma}\sinh\left(\frac{\gamma(T-t)}{2}\right)\right)^{\frac{2ab}{\sigma^2}}}. \tag{22.14}$$

We can now substitute our expression for $\exp\{\alpha(t)y\}$ given by (22.12) and our expression for $\exp\{\beta(t)\}$ given by (22.14) into our guess for $f(t,x,y)$ given by (22.2) to conclude

$$\begin{aligned} f(t,x,y) &= \exp\{\alpha(t)y + \beta(t)\}\exp\{i\theta x\} \\ &= \frac{\exp\left\{i\theta x - \frac{(i\theta+\theta^2)y}{\gamma\coth\left(\frac{\gamma(T-t)}{2}\right)+(a-i\sigma\rho\theta)} + \frac{ab(a-i\sigma\rho\theta)(T-t)}{\sigma^2} + i\theta\mu(T-t)\right\}}{\left(\cosh\left(\frac{\gamma(T-t)}{2}\right) + \frac{a-i\sigma\rho\theta}{\gamma}\sinh\left(\frac{\gamma(T-t)}{2}\right)\right)^{\frac{2ab}{\sigma^2}}}. \end{aligned}$$

Taking $t = 0$ gives

$$\varphi_{X_T}(\theta) = f(0,x,y) = \frac{\exp\left\{i\theta x - \frac{(i\theta+\theta^2)y}{\gamma\coth\frac{\gamma T}{2}+(a-i\sigma\rho\theta)} + \frac{abT(a-i\sigma\rho\theta)}{\sigma^2} + i\theta\mu T\right\}}{\left(\cosh\frac{\gamma T}{2} + \frac{a-i\sigma\rho\theta}{\gamma}\sinh\frac{\gamma T}{2}\right)^{\frac{2ab}{\sigma^2}}}.$$

and we are done!