Lecture #33, 34: The Characteristic Function for a Diffusion

Recall that the characteristic function of a random variable X is the function $\varphi_X : \mathbb{R} \to \mathbb{C}$ defined by $\varphi_X(\theta) = \mathbb{E}(e^{i\theta X})$. From Exercise 3.9, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the characteristic function of X is

$$\varphi_X(\theta) = \exp\left\{ i\mu\theta - \frac{\sigma^2\theta^2}{2} \right\}.$$

Suppose that $\{X_t, t \geq 0\}$ is a stochastic process. For each $T \geq 0$, we know that X_T is a random variable. Thus, we can consider $\varphi_{X_T}(\theta)$.

In the particular case that $\{X_t, t \geq 0\}$ is a diffusion defined by the stochastic differential equation

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}B_t + \mu(t, X_t)\,\mathrm{d}t \tag{21.1}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, if we can solve the SDE, then we can determine $\varphi_{X_T}(\theta)$ for any $T \geq 0$.

Example 21.1. Consider the case when both coefficients in (21.1) are constant so that

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + \mu\,\mathrm{d}t$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, In this case, the SDE is trivial to solve. If $X_0 = x$ is constant, then for any $T \geq 0$, we have

$$X_T = x + \sigma B_T + \mu T$$

which is simply arithmetic Brownian motion started at x. Therefore,

$$X_T \sim \mathcal{N}(x + \mu T, \sigma^2 T)$$

so that

$$\varphi_{X_T}(\theta) = \exp\left\{ i(x + \mu T)\theta - \frac{\sigma^2 T\theta^2}{2} \right\}.$$

Example 21.2. Consider the Ornstein-Uhlenbeck stochastic differential equation given by

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + aX_t\,\mathrm{d}t$$

where σ and a are constants. As we saw in Lecture #31, if $X_0 = x$ is constant, then for any $T \geq 0$, we have

$$X_T = e^{aT}x + \sigma\int_0^T e^{a(T-s)}\,\mathrm{d}B_s \sim \mathcal{N}\left(xe^{aT}, \frac{\sigma^2(e^{2aT}-1)}{2a} \right).$$

Therefore,

$$\varphi_{X_T}(\theta) = \exp\left\{ ixe^{aT}\theta - \frac{\sigma^2(e^{2aT}-1)\theta^2}{4a} \right\}.$$

Now it might seem like the only way to determine the characteristic function $\varphi_{X_T}(\theta)$ if $\{X_t, t \geq 0\}$ is a diffusion defined by (21.1) is to solve this SDE. Fortunately, this is not true. In many cases, the characteristic function for a diffusion defined by a SDE can be found using the *Feynman-Kac representation theorem* without actually solving the SDE.

Consider the diffusion

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}B_t + \mu(t, X_t)\,\mathrm{d}t. \tag{21.2}$$

We know from Version IV of Itô's formula (Theorem 14.12) that if $f \in C^1([0,\infty)) \times C^2(\mathbb{R})$, then

$$\begin{aligned}\mathrm{d}f(t, X_t) &= f'(t, X_t)\,\mathrm{d}X_t + \frac{1}{2}f''(t, X_t)\,\mathrm{d}\langle X\rangle_t + \dot{f}(t, x)\,\mathrm{d}t \\ &= \sigma(t, X_t)f'(t, X_t)\,\mathrm{d}B_t + \left[\mu(t, X_t)f'(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)f''(t, X_t) + \dot{f}(t, x)\right]\mathrm{d}t.\end{aligned}$$

We also know from Theorem 12.6 that any Itô integral is a martingale. Therefore, if we can find a particular function $f(t, x)$ such that the $\mathrm{d}t$ term is zero, then $f(t, X_t)$ will be a martingale. We define the *differential operator* (sometimes called the *generator* of the diffusion) to be the operator $\mathcal{A}$ given by

$$(\mathcal{A}f)(t, x) = \mu(t, x)f'(t, x) + \frac{1}{2}\sigma^2(t, x)f''(t, x) + \dot{f}(t, x).$$

Note that Version IV of Itô's formula now takes the form

$$\mathrm{d}f(t, X_t) = \sigma(t, X_t)f'(t, X_t)\,\mathrm{d}B_t + (\mathcal{A}f)(t, X_t)\,\mathrm{d}t.$$

This shows us the first connection between stochastic calculus and differential equations, namely that if $\{X_t, t \geq 0\}$ is a diffusion defined by (21.2) and if $f \in C^1([0,\infty)) \times C^2(\mathbb{R})$, then $f(t, X_t)$ is a martingale if and only if f satisfies the partial differential equation

$$(\mathcal{A}f)(t, x) = 0.$$

The Feynman-Kac representation theorem extends this idea by providing an explicit formula for the solution of this partial differential equation subject to certain boundary conditions.

Theorem 21.3 (Feynman-Kac Representation Theorem)**.** *Suppose that $u \in C^2(\mathbb{R})$, and let $\{X_t, t \geq 0\}$ be defined by the SDE*

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}B_t + \mu(t, X_t)\,\mathrm{d}t.$$

The unique bounded function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying the partial differential equation

$$(\mathcal{A}f)(t, x) = \mu(t, x)f'(t, x) + \frac{1}{2}\sigma^2(t, x)f''(t, x) + \dot{f}(t, x) = 0, \quad 0 \leq t \leq T, \;\; x \in \mathbb{R},$$

subject to the terminal condition

$$f(T, x) = u(x), \quad x \in \mathbb{R},$$

is given by

$$f(t, x) = \mathbb{E}[u(X_T)|X_t = x].$$

Example 21.4. We will now use the Feynman-Kac representation theorem to derive the characteristic function for arithmetic Brownian motion satisfying the SDE

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + \mu\,\mathrm{d}t$$

where σ, μ, and $X_0 = x$ are constants. Let $u(x) = e^{i\theta x}$ so that the Feynman-Kac representation theorem implies

$$f(t,x) = \mathbb{E}[u(X_T)|X_t = x] = \mathbb{E}[e^{i\theta X_T}|X_t = x]$$

is the unique bounded solution of the partial differential equation

$$\mu f'(t,x) + \frac{1}{2}\sigma^2 f''(t,x) + \dot{f}(t,x) = 0, \quad 0 \le t \le T, \;\; x \in \mathbb{R}, \tag{21.3}$$

subject to the terminal condition

$$f(T,x) = e^{i\theta x}, \quad x \in \mathbb{R}.$$

Note that $f(0,x) = \mathbb{E}[e^{i\theta X_T}|X_0 = x] = \varphi_{X_T}(\theta)$ is the characteristic function of X_T.

In order to solve (21.3) we use separation of variables. That is, we guess that $f(t,x)$ can be written as a function of x only times a function of t only so that

$$f(t,x) = \chi(x)\tau(t), \quad 0 \le t \le T, \;\; x \in \mathbb{R}. \tag{21.4}$$

Therefore, we find

$$f'(t,x) = \chi'(x)\tau(t), \quad f''(t,x) = \chi''(x)\tau(t), \quad \dot{f}(t,x) = \chi(x)\tau'(t)$$

so that (21.3) implies

$$\mu\chi'(x)\tau(t) + \frac{1}{2}\sigma^2\chi''(x)\tau(t) + \chi(x)\tau'(t) = 0, \quad 0 \le t \le T, \;\; x \in \mathbb{R},$$

or equivalently,

$$\frac{\mu\chi'(x)}{\chi(x)} + \frac{\sigma^2\chi''(x)}{2\chi(x)} = -\frac{\tau'(t)}{\tau(t)}.$$

Since the left side of this equation which is a function of x only equals the right side which is a function of t only, we conclude that both sides must be constant. For ease, we will write the constant as $-\lambda^2$. Thus, we must solve the two ordinary differential equations

$$\frac{\mu\chi'(x)}{\chi(x)} + \frac{\sigma^2\chi''(x)}{2\chi(x)} = -\lambda^2 \quad \text{and} \quad -\frac{\tau'(t)}{\tau(t)} = -\lambda^2.$$

The ODE for τ is easy to solve; clearly $\tau'(t) = \lambda^2\tau(t)$ implies

$$\tau(t) = C\exp\{\lambda^2 t\}$$

where C is an arbitrary constant. The ODE for χ is

$$\mu\chi'(x) + \frac{\sigma^2\chi''(x)}{2} = -\lambda^2\chi(x),$$

or equivalently,

$$\sigma^2\chi''(x) + 2\mu\chi'(x) + 2\lambda^2\chi(x) = 0. \tag{21.5}$$

Although this ODE is reasonably straightforward to solve for χ, it turns out that we do not need to actually solve it. This is because of our terminal condition. We know that

$$f(T, x) = e^{i\theta x}$$

and we also have assumed that

$$f(t, x) = \chi(x)\tau(t).$$

This implies that

$$f(T, x) = \chi(x)\tau(T) = e^{i\theta x}$$

which means that

$$\tau(T) = 1 \quad \text{and} \quad \chi(x) = e^{i\theta x}.$$

We now realize that we can solve for the arbitrary constant C; that is,

$$\tau(t) = C\exp\{\lambda^2 t\} \quad \text{and} \quad \tau(T) = 1$$

implies

$$C = \exp\{-\lambda^2 T\} \quad \text{so that} \quad \tau(t) = \exp\{-\lambda^2(T-t)\}.$$

We are also in a position to determine the value of λ^2. That is, we know that $\chi(x) = e^{i\theta x}$ must be a solution to the ODE (21.5). Thus, we simply need to choose λ^2 so that this is true. Since

$$\chi'(x) = i\theta e^{i\theta x} \quad \text{and} \quad \chi''(x) = -\theta^2 e^{i\theta x},$$

we conclude that

$$-\sigma^2\theta^2 e^{i\theta x} + 2i\mu\theta e^{i\theta x} + 2\lambda^2 e^{i\theta x} = 0,$$

and so factoring out $e^{i\theta x}$ gives

$$-\sigma^2\theta^2 + 2i\mu\theta + 2\lambda^2 = 0.$$

Thus,

$$-\lambda^2 = i\mu\theta - \frac{\sigma^2\theta^2}{2}$$

so that substituting in for $\tau(t)$ gives

$$\tau(t) = \exp\left\{i\mu(T-t)\theta - \frac{\sigma^2(T-t)\theta^2}{2}\right\}$$

and so from (21.4) we conclude

$$\begin{aligned} f(t, x) = \chi(x)\tau(t) &= e^{i\theta x}\exp\left\{i\mu(T-t)\theta - \frac{\sigma^2(T-t)\theta^2}{2}\right\} \\ &= \exp\left\{i(x + \mu(T-t))\theta - \frac{\sigma^2(T-t)\theta^2}{2}\right\}. \end{aligned}$$

Taking $t = 0$ gives

$$\varphi_{X_T}(\theta) = f(0, x) = \exp\left\{i(x + \mu T)\theta - \frac{\sigma^2 T\theta^2}{2}\right\}$$

in agreement with Example 21.1.

Example 21.5. We will now use the Feynman-Kac representation theorem to derive the characteristic function for a process satisfying the Ornstein-Uhlenbeck SDE

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + aX_t\,\mathrm{d}t$$

where σ, a, and $X_0 = x$ are constants. Let $u(x) = e^{i\theta x}$ so that the Feynman-Kac representation theorem implies

$$f(t,x) = \mathbb{E}[u(X_T)|X_t = x] = \mathbb{E}[e^{i\theta X_T}|X_t = x]$$

is the unique bounded solution of the partial differential equation

$$axf'(t,x) + \frac{1}{2}\sigma^2 f''(t,x) + \dot{f}(t,x) = 0, \quad 0 \le t \le T, \;\; x \in \mathbb{R}, \tag{21.6}$$

subject to the terminal condition

$$f(T,x) = e^{i\theta x}, \quad x \in \mathbb{R}.$$

Note that $f(0,x) = \mathbb{E}[e^{i\theta X_T}|X_0 = x] = \varphi_{X_T}(\theta)$ is the characteristic function of X_T.

If we try to use separation of variables to solve (21.6), then we soon discover that it does not produce a solution. Thus, we are forced to conclude that the solution $f(t,x)$ is not separable and is necessarily more complicated. Guided by the form of the terminal condition, we guess that $f(t,x)$ can be written as

$$f(t,x) = \exp\{i\theta\alpha(t)x + \beta(t)\}, \quad 0 \le t \le T, \;\; x \in \mathbb{R}, \tag{21.7}$$

for some functions $\alpha(t)$ and $\beta(t)$ of t only satisfying $\alpha(T) = 1$ and $\beta(T) = 0$. Differentiating we find

$$\begin{aligned} f'(t,x) &= i\theta\alpha(t)\exp\{i\theta\alpha(t)x + \beta(t)\} = i\theta\alpha(t)f(t,x), \\ f''(t,x) &= -\theta^2\alpha^2(t)\exp\{i\theta\alpha(t)x + \beta(t)\} = -\theta^2\alpha^2(t)f(t,x), \quad \text{and} \\ \dot{f}(t,x) &= [i\theta\alpha'(t)x + \beta'(t)]\exp\{i\theta\alpha(t)x + \beta(t)\} = [i\theta\alpha'(t)x + \beta'(t)]f(t,x) \end{aligned}$$

so that (21.6) implies

$$i\theta ax\alpha(t)f(t,x) - \frac{\sigma^2\theta^2}{2}\alpha^2(t)f(t,x) + [i\theta\alpha'(t)x + \beta'(t)]f(t,x) = 0, \quad 0 \le t \le T, \;\; x \in \mathbb{R}.$$

Factoring out the common $f(t,x)$ reduces the equation to

$$i\theta ax\alpha(t) - \frac{\sigma^2\theta^2}{2}\alpha^2(t) + i\theta\alpha'(t)x + \beta'(t) = 0,$$

or equivalently,

$$i\theta[a\alpha(t) + \alpha'(t)]x + \beta'(t) - \frac{\sigma^2\theta^2}{2}\alpha^2(t) = 0.$$

Since this equation must be true for all $0 \le t \le T$ and $x \in \mathbb{R}$, the only way that is possible is if the coefficient of x is zero *and* the constant term is 0. Thus, we must have

$$a\alpha(t) + \alpha'(t) = 0 \quad \text{and} \quad \beta'(t) - \frac{\sigma^2\theta^2}{2}\alpha^2(t) = 0. \tag{21.8}$$

This first equation in (21.8) involves only $\alpha(t)$ and is easily solved. That is, $\alpha'(t) = -a\alpha(t)$ implies $\alpha(t) = Ce^{-at}$ for some arbitrary constant C. The terminal condition $\alpha(T) = 1$ implies that $C = e^{aT}$ so that

$$\alpha(t) = e^{a(T-t)}.$$

Since we have solved for $\alpha(t)$, we can now solve the second equation in (21.8); that is,

$$\beta'(t) = \frac{\sigma^2\theta^2}{2}\alpha^2(t) = \frac{\sigma^2\theta^2}{2}e^{2a(T-t)}.$$

We simply integrate from 0 to t to find $\beta(t)$:

$$\beta(t) - \beta(0) = \frac{\sigma^2\theta^2}{2}\int_0^t e^{2a(T-s)}\,\mathrm{d}s = \frac{\sigma^2\theta^2}{4a}(e^{2aT} - e^{2a(T-t)}).$$

The terminal condition $\beta(T) = 0$ implies that

$$\beta(0) = \frac{\sigma^2\theta^2}{4a}(1 - e^{2aT})$$

and so

$$\beta(t) = \frac{\sigma^2\theta^2}{4a}(1 - e^{2aT}) + \frac{\sigma^2\theta^2}{4a}(e^{2aT} - e^{2a(T-t)}) = \frac{\sigma^2(1 - e^{2a(T-t)})\theta^2}{4a} = -\frac{\sigma^2(e^{2a(T-t)} - 1)\theta^2}{4a}.$$

Thus, from (21.7) we are now able to conclude that

$$f(t,x) = \exp\{i\theta\alpha(t)x + \beta(t)\} = \exp\left\{i\theta e^{a(T-t)}x - \frac{\sigma^2(e^{2a(T-t)} - 1)\theta^2}{4a}\right\}$$

for $0 \le t \le T$ and $x \in \mathbb{R}$. Taking $t = 0$ gives

$$\varphi_{X_T}(\theta) = f(0,x) = \exp\left\{i\theta e^{aT}x - \frac{\sigma^2(e^{2aT} - 1)\theta^2}{4a}\right\}$$

in agreement with Example 21.2.