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Lecture #31, 32: The Ornstein-Uhlenbeck Process as a Model of Volatility

The Ornstein-Uhlenbeck process is a diffusion process that was introduced as a model of the velocity of a particle undergoing Brownian motion. We know from Newtonian physics that the velocity of a (classical) particle in motion is given by the time derivative of its position. However, if the position of a particle is described by Brownian motion, then the time derivative does not exist. The Ornstein-Uhlenbeck process is an attempt to overcome this difficulty by modelling the velocity directly. Furthermore, just as Brownian motion is the scaling limit of simple random walk, the Ornstein-Uhlenbeck process is the scaling limit of the Ehrenfest urn model which describes the diffusion of particles through a permeable membrane.

In recent years, however, the Ornstein-Uhlenbeck process has appeared in finance as a model of the volatility of the underlying asset price process.

Suppose that the price of a stock $\{S_t, t \ge 0\}$ is modelled by geometric Brownian motion with volatility σ and drift μ so that S_t satisfies the SDE

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}B_t + \mu S_t \,\mathrm{d}t.$$

However, market data indicates that implied volatilities for different strike prices and expiry dates of options are not constant. Instead, they appear to be *smile* shaped (or *frown* shaped).

Perhaps the most natural approach is to allow for the volatility $\sigma(t)$ to be a deterministic function of time so that S_t satisfies the SDE

$$\mathrm{d}S_t = \sigma(t) S_t \,\mathrm{d}B_t + \mu S_t \,\mathrm{d}t.$$

This was already suggested by Merton in 1973. Although it does explain the different implied volatility levels for different expiry dates, it does not explain the smile shape for different strike prices.

Instead, Hull and White in 1987 proposed to use a stochastic volatility model where the underlying stock price $\{S_t, t \ge 0\}$ satisfies the SDE

$$\mathrm{d}S_t = \sqrt{v_t}\, S_t \,\mathrm{d}B_t + \mu S_t \,\mathrm{d}t$$

and the variance process $\{v_t, t \ge 0\}$ is given by geometric Brownian motion

$$\mathrm{d}v_t = c_1 v_t \,\mathrm{d}B_t + c_2 v_t \,\mathrm{d}t$$

with c_1 and c_2 known constants. The problem with this model is that geometric Brownian motion tends to increase exponentially which is an undesirable property for volatility.

Market data also indicates that volatility exhibits mean-reverting behaviour. This lead Stein and Stein in 1991 to introduce the mean-reverting Ornstein-Uhlenbeck process satisfying

$$\mathrm{d}v_t = \sigma \,\mathrm{d}B_t + a(b - v_t) \,\mathrm{d}t$$

where a, b, and σ are known constants. This process, however, allows negative values of v_t.

In 1993 Heston overcame this difficulty by considering a more complex stochastic volatility model. Before investigating the Heston model, however, we will consider the Ornstein-Uhlenbeck process separately and prove that negative volatilities are allowed thereby verifying that the Stein and Stein stock price model is flawed.

We say that the process $\{X_t, t \geq 0\}$ is an *Ornstein-Uhlenbeck process* if X_t satisfies the *Ornstein-Uhlenbeck stochastic differential equation* given by

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + aX_t\,\mathrm{d}t \tag{20.1}$$

where σ and a are constants and $\{B_t, t \geq 0\}$ is a standard Brownian motion.

Remark. Sometimes (20.1) is called the *Langevin equation*, especially in physics contexts.

Remark. The Ornstein-Uhlenbeck SDE is very similar to the SDE for geometric Brownian motion; the only difference is the absence of X_t in the $\mathrm{d}B_t$ term of (20.1). However, this slight change makes (20.1) more challenging to solve.

The "trick" for solving (20.1) is to multiply both sides by the *integrating factor* e^{-at} and to compare with $\mathrm{d}(e^{-at}X_t)$. The chain rule tells us that

$$\mathrm{d}(e^{-at}X_t) = e^{-at}\,\mathrm{d}X_t + X_t\,\mathrm{d}(e^{-at}) = e^{-at}\,\mathrm{d}X_t - ae^{-at}X_t\,\mathrm{d}t \tag{20.2}$$

and multiplying (20.1) by e^{-at} gives

$$e^{-at}\mathrm{d}X_t = \sigma e^{-at}\,\mathrm{d}B_t + ae^{-at}X_t\,\mathrm{d}t \tag{20.3}$$

so that substituting (20.3) into (20.2) gives

$$\mathrm{d}(e^{-at}X_t) = \sigma e^{-at}\,\mathrm{d}B_t + ae^{-at}X_t\,\mathrm{d}t - ae^{-at}X_t\,\mathrm{d}t = \sigma e^{-at}\,\mathrm{d}B_t.$$

Since $\mathrm{d}(e^{-at}X_t) = \sigma e^{-at}\mathrm{d}B_t$, we can now integrate to conclude that

$$e^{-at}X_t - X_0 = \sigma \int_0^t e^{-as}\,\mathrm{d}B_s$$

and so

$$X_t = e^{at}X_0 + \sigma \int_0^t e^{a(t-s)}\,\mathrm{d}B_s. \tag{20.4}$$

Observe that the integral in (20.4) is a Wiener integral. Definition 8.1 tells us that

$$\int_0^t e^{a(t-s)}\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t e^{2a(t-s)}\,\mathrm{d}s\right) = \mathcal{N}\left(0, \frac{e^{2at}-1}{2a}\right).$$

In particular, choosing $X_0 = x$ to be constant implies that

$$X_t = e^{at}x + \sigma \int_0^t e^{a(t-s)}\,\mathrm{d}B_s \sim \mathcal{N}\left(xe^{at}, \frac{\sigma^2(e^{2at}-1)}{2a}\right).$$

Actually, we can generalize this slightly. If we choose $X_0 \sim \mathcal{N}(x, \tau^2)$ independently of $\{B_t, t \geq 0\}$, then Exercise 3.12 tells us that

$$X_t = e^{at} X_0 + \sigma \int_0^t e^{a(t-s)} \, \mathrm{d}B_s \sim \mathcal{N}\left(xe^{at}, \tau^2 e^{2at} + \frac{\sigma^2 (e^{2at} - 1)}{2a} \right)$$
$$= \mathcal{N}\left(xe^{at}, \left(\tau^2 + \frac{\sigma^2}{2a} \right) e^{2at} - \frac{\sigma^2}{2a} \right).$$

Exercise 20.1. Suppose that $\{X_t, t \geq 0\}$ is an Ornstein-Uhlenbeck process given by (20.4) with $X_0 = 0$. If $s < t$, compute $\operatorname{Cov}(X_s, X_t)$.

We say that the process $\{X_t, t \geq 0\}$ is a *mean-reverting Ornstein-Uhlenbeck process* if X_t satisfies the SDE

$$\mathrm{d}X_t = \sigma \, \mathrm{d}B_t + (b - X_t) \, \mathrm{d}t \tag{20.5}$$

where σ and b are constants and $\{B_t, t \geq 0\}$ is a standard Brownian motion.

The trick for solving the mean-reverting Ornstein-Uhlenbeck process is similar. That is, we multiply by e^t and compare with $\mathrm{d}(e^t(b - X_t))$. The chain rule tells us that

$$\mathrm{d}(e^t(b - X_t)) = -e^t \, \mathrm{d}X_t + e^t(b - X_t) \, \mathrm{d}t \tag{20.6}$$

and multiplying (20.5) by e^t gives

$$e^t \mathrm{d}X_t = \sigma e^t \, \mathrm{d}B_t + e^t(b - X_t) \, \mathrm{d}t \tag{20.7}$$

so that substituting (20.7) into (20.6) gives

$$\mathrm{d}(e^t(b - X_t)) = -\sigma e^t \, \mathrm{d}B_t - e^t(b - X_t) \, \mathrm{d}t + e^t(b - X_t) \, \mathrm{d}t = -\sigma e^t \, \mathrm{d}B_t.$$

Since $\mathrm{d}(e^t(b - X_t)) = -\sigma e^t \, \mathrm{d}B_t$, we can now integrate to conclude that

$$e^t(b - X_t) - (b - X_0) = -\sigma \int_0^t e^s \, \mathrm{d}B_s$$

and so

$$X_t = (1 - e^{-t})b + e^{-t} X_0 + \sigma \int_0^t e^{s-t} \, \mathrm{d}B_s. \tag{20.8}$$

Exercise 20.2. Suppose that $X_0 \sim \mathcal{N}(x, \tau^2)$ is independent of $\{B_t, t \geq 0\}$. Determine the distribution of X_t given by (20.8).

Exercise 20.3. Use an appropriate integrating factor to solve the mean-reverting Ornstein-Uhlenbeck SDE considered by Stein and Stein, namely

$$\mathrm{d}X_t = \sigma \, \mathrm{d}B_t + a(b - X_t) \, \mathrm{d}t.$$

Assuming that $X_0 = x$ is constant, determine the distribution of X_t and conclude that $\mathbf{P}\{X_t < 0\} > 0$ for every $t > 0$. *Hint: X_t has a normal distribution.* This then explains our earlier claim that the Stein and Stein model is flawed.

As previous noted, Heston introduced a stochastic volatility model in 1993 that overcame this difficulty. Assume that the asset price process $\{S_t, t \geq 0\}$ satisfies the SDE

$$\mathrm{d}S_t = \sqrt{v_t}\, S_t \,\mathrm{d}B_t^{(1)} + \mu S_t \,\mathrm{d}t$$

where the variance process $\{v_t, t \geq 0\}$ satisfies

$$\mathrm{d}v_t = \sigma\sqrt{v_t}\,\mathrm{d}B_t^{(2)} + a(b - v_t)\,\mathrm{d}t \tag{20.9}$$

and the two driving Brownian motions $\{B_t^{(1)}, t \geq 0\}$ and $\{B_t^{(2)}, t \geq 0\}$ are correlated with rate ρ, i.e.,

$$\mathrm{d}\langle B^{(1)}, B^{(2)}\rangle_t = \rho\,\mathrm{d}t.$$

The $\sqrt{v_t}$ term in (20.9) is needed to guarantee positive volatility—when the process touches zero the stochastic part becomes zero and the non-stochastic part will push it up. The parameter a measures the speed of the mean-reversion, b is the average level of volatility, and σ is the volatility of volatility. Market data suggests that the correlation rate ρ is typically negative. The negative dependence between returns and volatility is sometimes called the *leverage effect*.

Heston's model involves a system of stochastic differential equations. The key tool for analyzing such a system is the multidimensional version of Itô's formula.

Theorem 20.4 (Version V)**.** *Suppose that $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are diffusions defined by the stochastic differential equations*

$$\mathrm{d}X_t = a_1(t, X_t, Y_t)\,\mathrm{d}B_t^{(1)} + b_1(t, X_t, Y_t)\,\mathrm{d}t$$

and

$$\mathrm{d}Y_t = a_2(t, X_t, Y_t)\,\mathrm{d}B_t^{(2)} + b_2(t, X_t, Y_t)\,\mathrm{d}t,$$

respectively, where $\{B_t^{(1)}, t \geq 0\}$ and $\{B_t^{(2)}, t \geq 0\}$ are each standard one-dimensional Brownian motions. If $f \in C^1([0,\infty)) \times C^2(\mathbb{R}^2)$, then

$$\begin{aligned}\mathrm{d}f(t, X_t, Y_t) = \dot{f}(t, X_t, Y_t)\,\mathrm{d}t + f_1(t, X_t, Y_t)\,\mathrm{d}X_t + \frac{1}{2}f_{11}(t, X_t, Y_t)\,\mathrm{d}\langle X\rangle_t \\ + f_2(t, X_t, Y_t)\,\mathrm{d}Y_t + \frac{1}{2}f_{22}(t, X_t, Y_t)\,\mathrm{d}\langle Y\rangle_t + f_{12}(t, X_t, Y_t)\,\mathrm{d}\langle X, Y\rangle_t\end{aligned}$$

where the partial derivatives are defined as

$$\dot{f}(t,x,y) = \frac{\partial}{\partial t}f(t,x,y), \quad f_1(t,x,y) = \frac{\partial}{\partial x}f(t,x,y), \quad f_{11}(t,x,y) = \frac{\partial^2}{\partial x^2}f(t,x,y)$$

$$f_2(t,x,y) = \frac{\partial}{\partial y}f(t,x,y), \quad f_{22}(t,x,y) = \frac{\partial^2}{\partial y^2}f(t,x,y), \quad f_{12}(t,x,y) = \frac{\partial^2}{\partial x \partial y}f(t,x,y),$$

and $\mathrm{d}\langle X, Y\rangle_t$ is computed according to the rule

$$\mathrm{d}\langle X, Y\rangle_t = (\mathrm{d}X_t)(\mathrm{d}Y_t) = a_1(t, X_t, Y_t)a_2(t, X_t, Y_t)\,\mathrm{d}\langle B^{(1)}, B^{(2)}\rangle_t.$$

Remark. In a typical problem involving the multidimensional version of Itô's formula, the quadratic covariation process $\langle B^{(1)}, B^{(2)} \rangle_t$ will be specified. However, two particular examples are worth mentioning. If $B^{(1)} = B^{(2)}$, then $\mathrm{d}\langle B^{(1)}, B^{(2)} \rangle_t = \mathrm{d}t$, whereas if $B^{(1)}$ and $B^{(2)}$ are independent, then $\mathrm{d}\langle B^{(1)}, B^{(2)} \rangle_t = 0$.

Exercise 20.5. Suppose that $f(t, x, y) = xy$. Using Version V of Itô's formula (Theorem 20.4), verify that the *product rule for diffusions* is given by

$$\mathrm{d}(X_t Y_t) = X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + \mathrm{d}\langle X, Y \rangle_t.$$

Thus, our goal in the next few lectures is to price a European call option assuming that the underlying stock price follows Heston's model of geometric Brownian motion with a stochastic volatility, namely

$$\begin{cases} \mathrm{d}S_t = \sqrt{v_t}\, S_t \, \mathrm{d}B_t^{(1)} + \mu S_t \, \mathrm{d}t, \\ \mathrm{d}v_t = \sigma \sqrt{v_t} \, \mathrm{d}B_t^{(2)} + a(b - v_t) \, \mathrm{d}t, \\ \mathrm{d}\langle B^{(1)}, B^{(2)} \rangle_t = \rho \, \mathrm{d}t. \end{cases}$$