Statistics 441 (Fall 2014) November 17, 2014
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Lecture #30: Implied Volatility

Recall that if $V(0, S_0)$ denotes the fair price (at time 0) of a European call option with strike price E and expiry date T, then the Black-Scholes option valuation formula is

$$\begin{aligned} V(0, S_0) &= S_0\, \Phi\left(\frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ee^{-rT}\, \Phi\left(\frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &= S_0\, \Phi(d_1) - Ee^{-rT}\, \Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Suppose that at time 0 a first investor buys a European call option on the stock having initial price S_0 with strike price E and expiry date T. Of course, the fair price for the first investor to pay is $V(0, S_0)$.

Suppose that some time later, say at time t, a second investor wants to buy a European call option on the same stock with the same strike price E and the same expiry date T. What is the fair price for this second investor to pay at time t?

Since it is now time t, the value of the underlying stock, namely S_t, is known. The expiry date T is time $T - t$ away. Thus, we simply re-scale our original Black-Scholes solution so that t is the new time 0, the new initial price of the stock is S_t, and $T - t$ is the new expiry date. This implies that the fair price (at time t) of a European call option with strike price E and expiry date T is given by the Black-Scholes option valuation formula

$$\begin{aligned} &V(t, S_t) \\ &\quad = S_t\, \Phi\left(\frac{\log(S_t/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ee^{-r(T-t)}\, \Phi\left(\frac{\log(S_t/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad = S_t\, \Phi(d_1) - Ee^{-r(T-t)}\, \Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\log(S_t/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \text{ and } d_2 = \frac{\log(S_t/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

In fact, the rigorous justification for this is exactly the same as in (16.4) except now we view $M_t = W(t, S_t)$ as non-random since S_t, the stock price at time t, is known at time t. Note that the formula for $V(t, S_t)$ holds for $0 \le t \le T$. In particular, for $t = 0$ we recover our original result.

Remark. Given the general Black-Scholes formula $V(t, S_t)$, $0 \le t \le T$, we can define Θ as the derivative of V with respect to t. (In Lecture #29, we defined Θ as the derivative of V with respect to the expiry date T.) With this revised definition, we compute

$$\Theta = \frac{\partial V}{\partial t} = -Ere^{-r(T-t)}\,\Phi\left(d_2\right) - \frac{\sigma S_t}{2\sqrt{T-t}}\Phi'\left(d_1\right)$$

as in (10.5) of [11]. Note that there is a sign difference between this result and (18.3). All of the other Greeks, namely delta, gamma, rho, and vega, are the same as in Lecture #29 except that T is replaced with $T - t$.

The practical advantage of the Black-Scholes formula $V(t, S_t)$ is that it allows for the fast and easy calculation of option prices. It is worth noting, however, that "exact" calculations are not actually possible since the formula is given in terms of Φ, the normal cumulative distribution function. In order to "evaluate" $\Phi(d_1)$ or $\Phi(d_2)$ one must resort to using a computer (or table of normal values). Computationally, it is quite easy to evaluate Φ to many decimal places accuracy; and so this is the reason that we say the Black-Scholes formulation gives an exact formula. (In fact, programs like MATLAB or R can easily give values of Φ accurate to 10 decimal places.)

However, the limitations of the Black-Scholes model are numerous. Assumptions such as the asset price process following a geometric Brownian motion (so that an asset price at a fixed time has a lognormal distribution), or that the asset's volatility is constant, are not justified by actual market data.

As such, one of the goals of modern finance is to develop more sophisticated models for the asset price process, and to then develop the necessary stochastic calculus to produce a "solution" to the pricing problem. Unfortunately, there is no other model that produces as compact a solution as Black-Scholes. This means that the "solution" to any other model involves numerical analysis—and often quite involved analysis at that.

Suppose, for the moment, that we assume that the Black-Scholes model is valid. In particular, assume that the stock price $\{S_t, t \ge 0\}$ follows geometric Brownian motion. The fair price $V(t, S_t)$ to pay at time t depends on the parameters S_t, E, $T - t$, r, and σ^2. Of these, only the asset volatility σ cannot be directly observed.

There are two distinct approaches to extracting a value of σ from market data. The first is known as *implied volatility* and is obtained by using a quoted option value to recover σ. The second is known as *historical volatility* and is essentially maximum likelihood estimation of the parameter σ.

We will discuss only implied volatility. For ease, we will focus on the time $t = 0$ case. Suppose that S_0, E, T, and r are all known, and consider $V(0, S_0)$. Since we are assuming that only σ is unknown, we will emphasis this by writing $V(\sigma)$.

Thus, if we have a quoted value of the option price, say V^*, then we want to solve the equation $V(\sigma) = V^*$ for σ.

We will now show there is a unique solution to this equation which will be denoted by σ^* so that $V(\sigma^*) = V^*$.

To begin, note that we are only interested in positive volatilities so that $\sigma \in [0, \infty)$. Furthermore, $V(\sigma)$ is continuous on $[0, \infty)$ with

$$\lim_{\sigma \to \infty} V(\sigma) = S_0 \quad \text{and} \quad \lim_{\sigma \to 0^+} V(\sigma) = \max\{S_0 - Ee^{-rT}, 0\}. \tag{19.1}$$

Recall that from Lecture #29 that

$$\text{vega } = \frac{\partial V}{\partial \sigma} = V'(\sigma) = S_0 \sqrt{T}\, \Phi'(d_1).$$

Since

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

we immediately conclude that $V'(\sigma) > 0$.

The fact that $V(\sigma)$ is continuous on $[0, \infty)$ with $V'(\sigma) > 0$ implies that $V(\sigma)$ is strictly increasing on $[0, \infty)$. Thus, we see that $V(\sigma) = V^*$ has a solution if and only if

$$\max\{S_0 - Ee^{-rT}, 0\} \le V^* \le S_0 \tag{19.2}$$

and that if a solution exists, then it must be unique. The no arbitrage assumption (i.e., the put-call parity) implies that the condition (19.2) always holds.

We now calculate

$$V''(\sigma) = \frac{\partial^2 V}{\partial \sigma^2} = \frac{d_1 d_2}{\sigma} \frac{\partial V}{\partial \sigma} = \frac{d_1 d_2}{\sigma} V'(\sigma) \tag{19.3}$$

which shows that the only inflection point of $V(\sigma)$ on $[0, \infty)$ is at

$$\hat{\sigma} = \sqrt{2 \left| \frac{\log(S_0/E) + rT}{T} \right|}. \tag{19.4}$$

Notice that we can write

$$V''(\sigma) = \frac{T}{4\sigma^3}(\hat{\sigma}^4 - \sigma^4) V'(\sigma) \tag{19.5}$$

which implies that $V(\sigma)$ is convex (i.e., concave up) for $\sigma < \hat{\sigma}$ and concave (i.e., concave down) for $\sigma > \hat{\sigma}$.

Exercise 19.1. Verify (19.1), (19.2), (19.3), (19.4), and (19.5).

The consequence of all of this is that *Newton's method* will be globally convergent for a suitably chosen initial value. Recall that Newton's method tells us that in order to solve the equation $F(x) = 0$, we consider the sequence of iterates $x_0, x_1, x_2, \ldots$ where

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

If we define

$$x^* = \lim_{n \to \infty} x_n,$$

then $F(x^*) = 0$. Of course, there are assumptions needed to ensure that Newton's method converges and produces the correct solution.

If we now consider $F(\sigma) = V(\sigma) - V^*$, then we have already shown that the conditions needed to guarantee that Newton's method converges have been satisfied.

It can also be shown that

$$0 < \frac{\sigma_{n+1} - \sigma^*}{\sigma_n - \sigma^*} < 1$$

for all n which implies that the error in the approximation is strictly decreasing as n increases. Thus, if we choose $\sigma_0 = \hat{\sigma}$, then the error must always converge to 0. Moreover, it can be shown that the convergence is quadratic. Thus, choosing $\sigma_0 = \hat{\sigma}$ is a foolproof (and deterministic) way of starting Newton's method. We can then stop iterating when our error is within some pre-specified tolerance, say $< 10^{-8}$.

Remark. Computing implied volatility using Newton's method is rather easy to implement in MATLAB. See, for instance, the program `ch14.m` from Higham [11].

Consider obtaining data that reports the option price V^* for a variety of values of the strike price E while at the same time holds r, S_0, and T fixed. An example of such data is presented in Section 14.5 of [11]. If the Black-Scholes formula were valid, then the volatility would be the same for each strike price. That is, the graph of strike price vs. implied volatility would be a horizontal line passing through $\sigma = \sigma^*$.

However, in this example, and in numerous other examples, the implied volatility curve appears to bend in the shape of either a *smile* or a *frown*.

Remark. More sophisticated analyses of implied volatility involve data that reports the option price V^* for a variety of values of the strike price E and expiry dates T while at the same time holding r and S_0 fixed. This produces a graph of strike price vs. expiry date vs. implied volatility, and the result is an *implied volatility surface*. The functional data analysis needed to in this case requires a number of statistical tools including *principal components analysis*. If the Black-Scholes formula were valid, then the resulting implied volatility surface would be a plane. Market data, however, typically results in bowl-shaped or hat-shaped surfaces. For details, see [23] which is also freely available online.

This implies, of course, that the Black-Scholes formula is not a perfect description of the option values that arise in practice. Many attempts have been made to "fix" this by considering stock price models that do not have constant volatility. We will investigate some such models next lecture.