Lecture #29: The Greeks

Recall that if $V(0, S_0)$ denotes the fair price (at time 0) of a European call option with strike price E and expiry date T, then the Black-Scholes option valuation formula is

$$\begin{aligned} V(0, S_0) &= S_0 \, \Phi\left(\frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ee^{-rT} \, \Phi\left(\frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &= S_0 \, \Phi(d_1) - Ee^{-rT} \, \Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

We see that this formula depends on

- the initial price of the stock S_0,
- the expiry date T,
- the strike price E,
- the risk-free interest rate r, and
- the stock's volatility σ.

The partial derivatives of $V = V(0, S_0)$ with respect to these variables are extremely important in practice, and we will now compute them; for ease, we will write $S = S_0$. In fact, some of these partial derivatives are given special names and referred to collectively as "the Greeks":

- $\Delta = \dfrac{\partial V}{\partial S}$ (delta),
- $\Gamma = \dfrac{\partial^2 V}{\partial S^2} = \dfrac{\partial \Delta}{\partial S}$ (gamma),
- $\rho = \dfrac{\partial V}{\partial r}$ (rho),
- $\Theta = \dfrac{\partial V}{\partial T}$ (theta),
- vega $= \dfrac{\partial V}{\partial \sigma}$.

Note. Vega is not actually a Greek letter. Sometimes it is written as ν (which is the Greek letter nu).

Remark. On page 80 of [11], Higham changes from using $V(0, S_0)$ to denote the fair price at time 0 of a European call option with strike price E and expiry date T to using $C(0, S_0)$. Both notations seem to be widely used in the literature.

The financial use of each of "The Greeks" is as follows.

- Delta measures sensitivity to a small change in the price of the underlying asset.
- Gamma measures the rate of change of delta.
- Rho measures sensitivity to the applicable risk-free interest rate.
- Theta measures sensitivity to the passage of time. Sometimes the financial definition of Θ is
$$-\frac{\partial V}{\partial T}.$$
With this definition, if you are "long an option, then you are short theta."
- Vega measures sensitivity to volatility.

Apparently, there are even more "Greeks."

- Lambda, the percentage change in the option value per unit change in the underlying asset price, is given by
$$\lambda = \frac{1}{V}\frac{\partial V}{\partial S} = \frac{\partial \log V}{\partial S}.$$
- Vega gamma, or volga, measures second-order sensitivity to volatility and is given by
$$\frac{\partial^2 V}{\partial \sigma^2}.$$
- Vanna measures cross-sensitivity of the option value with respect to change in the underlying asset price and the volatility and is given by
$$\frac{\partial^2 V}{\partial S \partial \sigma} = \frac{\partial \Delta}{\partial \sigma}.$$
It is also the sensitivity of delta to a unit change in volatility.
- Delta decay, or charm, given by
$$\frac{\partial^2 V}{\partial S \partial T} = \frac{\partial \Delta}{\partial T},$$
measures time decay of delta. (This can be important when hedging a position over the weekend.)

- Gamma decay, or colour, given by
$$\frac{\partial^3 V}{\partial S^2 \partial T},$$
measures the sensitivity of the charm to the underlying asset price.

- Speed, given by
$$\frac{\partial^3 V}{\partial S^3},$$
measures third-order sensitivity to the underlying asset price.

In order to actually perform all of the calculations of the Greeks, we need to recall that

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Furthermore, we observe that

$$\log\left(\frac{S\Phi'(d_1)}{Ee^{-rT}\Phi'(d_2)}\right) = 0 \tag{18.1}$$

which implies that

$$S\Phi'(d_1) - Ee^{-rT}\Phi'(d_2) = 0. \tag{18.2}$$

Exercise 18.1. Verify (18.1) and deduce (18.2).

Since

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

we find

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}, \quad \frac{\partial d_1}{\partial r} = \frac{\sqrt{T}}{\sigma}, \quad \frac{\partial d_1}{\partial \sigma} = \frac{\sigma^2 T - \left[\log(S/E) + (r + \frac{1}{2}\sigma^2)T\right]}{\sigma^2\sqrt{T}} = -\frac{d_2}{\sigma}, \quad \text{and}$$

$$\frac{\partial d_1}{\partial T} = \frac{-\log(S/E) + (r + \frac{1}{2}\sigma^2)T}{2\sigma T^{3/2}}.$$

Furthermore, since

$$d_2 = d_1 - \sigma\sqrt{T},$$

we conclude

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}}, \quad \frac{\partial d_2}{\partial r} = \frac{\sqrt{T}}{\sigma}, \quad \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T} = -\frac{d_2}{\sigma} - \sqrt{T}, \quad \text{and}$$

$$\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{\sigma}{2\sqrt{T}} = \frac{-\log(S/E) + (r + \frac{1}{2}\sigma^2)T}{2\sigma T^{3/2}} - \frac{\sigma}{2\sqrt{T}} = \frac{-\log(S/E) + (r - \frac{1}{2}\sigma^2)T}{2\sigma T^{3/2}}.$$

- **Delta.** Since $V = S\,\Phi\,(d_1) - Ee^{-rT}\,\Phi\,(d_2)$, we find

$$\begin{aligned}\Delta = \frac{\partial V}{\partial S} &= \Phi\,(d_1) + S\,\frac{\partial \Phi\,(d_1)}{\partial S} - Ee^{-rT}\,\frac{\partial \Phi\,(d_2)}{\partial S} \\ &= \Phi\,(d_1) + S\,\Phi'(d_1)\,\frac{\partial d_1}{\partial S} - Ee^{-rT}\,\Phi'(d_2)\,\frac{\partial d_2}{\partial S} \\ &= \Phi\,(d_1) + \frac{\Phi'(d_1)}{\sigma\sqrt{T}} - Ee^{-rT}\,\frac{\Phi'(d_2)}{S\sigma\sqrt{T}} \\ &= \Phi\,(d_1) + \frac{1}{S\sigma\sqrt{T}}\left[S\Phi'(d_1) - Ee^{-rT}\Phi'(d_2)\right] \\ &= \Phi\,(d_1)\end{aligned}$$

where the last step follows from (18.2).

- **Gamma.** Since $\Delta = \Phi(d_1)$, we find

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \Phi'(d_1)\,\frac{\partial d_1}{\partial S} = \frac{\Phi'(d_1)}{S\sigma\sqrt{T}}.$$

- **Rho.** Since $V = S\,\Phi\,(d_1) - Ee^{-rT}\,\Phi\,(d_2)$, we find

$$\begin{aligned}\rho = \frac{\partial V}{\partial r} &= S\,\frac{\partial \Phi\,(d_1)}{\partial r} + ETe^{-rT}\,\Phi\,(d_2) - Ee^{-rT}\,\frac{\partial \Phi\,(d_2)}{\partial r} \\ &= S\,\Phi'\,(d_1)\,\frac{\partial d_1}{\partial r} + ETe^{-rT}\,\Phi\,(d_2) - Ee^{-rT}\,\Phi'\,(d_2)\,\frac{\partial d_2}{\partial r} \\ &= \frac{S\,\sqrt{T}}{\sigma}\,\Phi'\,(d_1) + ETe^{-rT}\,\Phi\,(d_2) - \frac{Ee^{-rT}\sqrt{T}}{\sigma}\,\Phi'\,(d_2) \\ &= \frac{\sqrt{T}}{\sigma}\left[S\Phi'(d_1) - Ee^{-rT}\Phi'(d_2)\right] + ETe^{-rT}\,\Phi\,(d_2) \\ &= ETe^{-rT}\,\Phi\,(d_2)\end{aligned}$$

where, as before, the last step follows from (18.2).

- **Theta.** Since $V = S\,\Phi\,(d_1) - Ee^{-rT}\,\Phi\,(d_2)$, we find

$$\begin{aligned}\Theta = \frac{\partial V}{\partial T} &= S\,\frac{\partial \Phi\,(d_1)}{\partial T} + Ere^{-rT}\,\Phi\,(d_2) - Ee^{-rT}\,\frac{\partial \Phi\,(d_2)}{\partial T} \\ &= S\,\Phi'\,(d_1)\,\frac{\partial d_1}{\partial T} + Ere^{-rT}\,\Phi\,(d_2) - Ee^{-rT}\,\Phi'\,(d_2)\,\frac{\partial d_2}{\partial T} \\ &= S\,\Phi'\,(d_1)\,\frac{\partial d_1}{\partial T} + Ere^{-rT}\,\Phi\,(d_2) - Ee^{-rT}\,\Phi'\,(d_2)\left[\frac{\partial d_1}{\partial T} - \frac{\sigma}{2\sqrt{T}}\right] \\ &= \left[S\,\Phi'\,(d_1) - Ee^{-rT}\,\Phi'\,(d_2)\right]\frac{\partial d_1}{\partial T} + Ere^{-rT}\,\Phi\,(d_2) + \frac{\sigma}{2\sqrt{T}}Ee^{-rT}\,\Phi'\,(d_2) \\ &= Ere^{-rT}\,\Phi\,(d_2) + \frac{\sigma}{2\sqrt{T}}Ee^{-rT}\,\Phi'\,(d_2)\end{aligned}$$

where, as before, the last step follows from (18.2). However, (18.2) also implies that we can write Θ as

$$\Theta = Ere^{-rT}\,\Phi\left(d_2\right) + \frac{\sigma S}{2\sqrt{T}}\Phi'\left(d_1\right). \tag{18.3}$$

- **Vega.** Since $V = S\,\Phi\left(d_1\right) - Ee^{-rT}\,\Phi\left(d_2\right)$, we find

$$\begin{aligned}
\text{vega } = \frac{\partial V}{\partial \sigma} &= S\,\frac{\partial \Phi\left(d_1\right)}{\partial \sigma} - Ee^{-rT}\,\frac{\partial \Phi\left(d_2\right)}{\partial \sigma} \\
&= S\,\Phi'\left(d_1\right)\,\frac{\partial d_1}{\partial \sigma} - Ee^{-rT}\,\Phi'\left(d_2\right)\,\frac{\partial d_2}{\partial \sigma} \\
&= -\frac{d_2}{\sigma}\,S\,\Phi'\left(d_1\right) - \left(-\frac{d_2}{\sigma} - \sqrt{T}\right)E\,e^{-rT}\,\Phi'\left(d_2\right) \\
&= -\frac{d_2}{\sigma}\left[S\,\Phi'\left(d_1\right) - Ee^{-rT}\,\Phi'\left(d_2\right)\right] + \sqrt{T}\,E\,e^{-rT}\,\Phi'\left(d_2\right) \\
&= \sqrt{T}\,E\,e^{-rT}\,\Phi'\left(d_2\right)
\end{aligned}$$

where, as before, the last step follows from (18.2). However, (18.2) also implies that we can write vega as

$$\text{vega } = S\,\sqrt{T}\,\Phi'\left(d_1\right).$$

Remark. Our definition of Θ is slightly different than the one in Higham [11]. We are differentiating V with respect to the expiry date T as opposed to an arbitrary time t with $0 \le t \le T$. This accounts for the discrepancy in the minus signs in (10.5) of [11] and (18.3).

Exercise 18.2. Compute lambda, volga, vanna, charm, colour, and speed for the Black-Scholes option valuation formula for a European call option with strike price E.

We also recall the put-call parity formula for European call and put options from Lecture #2:

$$V(0,S_0) + Ee^{-rT} = P(0,S_0) + S_0. \tag{18.4}$$

Here $P = P(0,S_0)$ is the fair price (at time 0) of a European put option with strike price E.

Exercise 18.3. Using the formula (18.4), compute the Greeks for a European put option. That is, compute

$$\Delta = \frac{\partial P}{\partial S}, \quad \Gamma = \frac{\partial^2 P}{\partial S^2}, \quad \rho = \frac{\partial P}{\partial r}, \quad \Theta = \frac{\partial P}{\partial T}, \quad \text{and} \quad \text{vega } = \frac{\partial P}{\partial \sigma}.$$

Note that gamma and vega for a European put option with strike price E are the same as gamma and vega for a European call option with strike price E.