Statistics 441 (Fall 2014) November 10, 2014
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Lecture #28: Calculations with Itô's Formula

Example 17.1 (Assignment #4, problem #10)**.** Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Determine an expression for

$$\int_0^t \sin(B_s)\,\mathrm{d}B_s$$

that does not involve Itô integrals.

Solution. Since Version I of Itô's formula tells us that

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s,$$

if we choose $f'(x) = \sin(x)$ so that $f(x) = -\cos(x)$ and $f''(x) = \cos(x)$, then

$$-\cos(B_t) + \cos(B_0) = \int_0^t \sin(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \cos(B_s)\,\mathrm{d}s.$$

The fact that $B_0 = 0$ implies

$$\int_0^t \sin(B_s)\,\mathrm{d}B_s = 1 - \cos(B_t) - \frac{1}{2}\int_0^t \cos(B_s)\,\mathrm{d}s.$$

Example 17.2 (Assignment #4, problem #1)**.** Suppose that $\{B_t, t \geq 0\}$ is a Brownian motion starting at 0. If the process $\{X_t, t \geq 0\}$ is defined by setting

$$X_t = \exp\{B_t\},$$

use Itô's formula to compute $\mathrm{d}X_t$.

Solution. Version I of Itô's formula tells us that

$$\mathrm{d}f(B_t) = f'(B_t)\,\mathrm{d}B_t + \frac{1}{2}f''(B_t)\,\mathrm{d}t$$

so that if $f(x) = e^x$, then

$$\mathrm{d}\exp\{B_t\} = \exp\{B_t\}\,\mathrm{d}B_t + \frac{1}{2}\exp\{B_t\}\,\mathrm{d}t.$$

Equivalently, if $X_t = \exp\{B_t\}$, then

$$\mathrm{d}X_t = X_t\,\mathrm{d}B_t + \frac{X_t}{2}\,\mathrm{d}t.$$

Example 17.3 (Assignment #4, problem #8). Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Consider the process $\{Y_t, t \geq 0\}$ defined by setting $Y_t = B_t^k$ where k is a positive integer. Use Itô's formula to show that Y_t satisfies the SDE

$$\mathrm{d}Y_t = kY_t^{1-1/k}\,\mathrm{d}B_t + \frac{k(k-1)}{2}Y_t^{1-2/k}\,\mathrm{d}t.$$

Solution. Version I of Itô's formula tells us that

$$\mathrm{d}f(B_t) = f'(B_t)\,\mathrm{d}B_t + \frac{1}{2}f''(B_t)\,\mathrm{d}t$$

so that if $f(x) = x^k$, then $f'(x) = kx^{k-1}$ and $f''(x) = k(k-1)x^{k-2}$ so that

$$\mathrm{d}B_t^k = kB_t^{k-1}\,\mathrm{d}B_t + \frac{k(k-1)}{2}B_t^{k-2}\,\mathrm{d}t.$$

Writing $Y_t = B_t^k$ gives

$$\mathrm{d}Y_t = kY_t^{1-1/k}\,\mathrm{d}B_t + \frac{k(k-1)}{2}Y_t^{1-2/k}\,\mathrm{d}t.$$

Example 17.4 (Assignment #4, problem #5). Consider the Itô process $\{Y_t, t \geq 0\}$ described by the stochastic differential equation

$$\mathrm{d}Y_t = 0.4\,\mathrm{d}B_t + 0.1\,\mathrm{d}t.$$

If the process $\{X_t, t \geq 0\}$ is defined by $X_t = e^{0.5Y_t}$, determine $\mathrm{d}X_t$.

Solution. Version III of Itô's formula tells us that

$$\mathrm{d}f(Y_t) = f'(Y_t)\,\mathrm{d}Y_t + \frac{1}{2}f''(Y_t)\,\mathrm{d}\langle Y\rangle_t$$

so that if $f(y) = e^{0.5y}$, then

$$\mathrm{d}\exp\{0.5Y_t\} = (0.5)\exp\{0.5Y_t\}\,\mathrm{d}Y_t + \frac{(0.5)^2}{2}\exp\{0.5Y_t\}\,\mathrm{d}\langle Y\rangle_t.$$

Since $\mathrm{d}Y_t = 0.4\,\mathrm{d}B_t + 0.1\,\mathrm{d}t$, we conclude that $\mathrm{d}\langle Y\rangle_t = (0.4)^2\,\mathrm{d}t = 0.16\,\mathrm{d}t$ and so

$$\mathrm{d}\exp\{0.5Y_t\} = (0.5)\exp\{0.5Y_t\}(0.4\,\mathrm{d}B_t + 0.1\,\mathrm{d}t) + \frac{(0.5)^2}{2}\exp\{0.5Y_t\}(0.16\,\mathrm{d}t).$$

Writing $X_t = e^{0.5Y_t}$ and collecting like terms gives

$$\mathrm{d}X_t = 0.2X_t\,\mathrm{d}B_t + 0.07X_t\,\mathrm{d}t.$$

Example 17.5 (Assignment #4, problem #11)**.** Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and suppose further that the process $\{X_t, t \geq 0\}$, $X_0 = a > 0$, satisfies the stochastic differential equation

$$\mathrm{d}X_t = X_t \,\mathrm{d}B_t + \frac{1}{X_t}\,\mathrm{d}t.$$

(a) If $f(x) = x^2$, determine $\mathrm{d}f(X_t)$.

(b) If $f(t, x) = t^2 x^2$, determine $\mathrm{d}f(t, X_t)$.

Solution. Version III of Itô's formula tells us that

$$\mathrm{d}f(X_t) = f'(X_t)\,\mathrm{d}X_t + \frac{1}{2} f''(X_t)\,\mathrm{d}\langle X\rangle_t$$

so that

$$\mathrm{d}(X_t^2) = 2X_t\,\mathrm{d}X_t + \mathrm{d}\langle X\rangle_t.$$

Version IV of Itô's formula tells us that

$$\mathrm{d}f(t, X_t) = \dot{f}(t, X_t)\,\mathrm{d}t + f'(t, X_t)\,\mathrm{d}X_t + \frac{1}{2} f''(t, X_t)\,\mathrm{d}\langle X\rangle_t$$

so that

$$\mathrm{d}(t^2 X_t^2) = 2tX_t^2\,\mathrm{d}t + 2t^2 X_t\,\mathrm{d}X_t + t^2\,\mathrm{d}\langle X\rangle_t.$$

Since

$$\mathrm{d}X_t = X_t\,\mathrm{d}B_t + \frac{1}{X_t}\,\mathrm{d}t,$$

we conclude that

$$\mathrm{d}\langle X\rangle_t = X_t^2\,\mathrm{d}t.$$

Thus,

(a) $\mathrm{d}(X_t^2) = 2X_t^2\,\mathrm{d}B_t + (2 + X_t^2)\,\mathrm{d}t$, and

(b) $\mathrm{d}(t^2 X_t^2) = 2t^2 X_t^2\,\mathrm{d}B_t + (2tX_t^2 + 2t^2 + t^2 X_t^2)\,\mathrm{d}t$.

Example 17.6 (Assignment #4, problem #7)**.** Suppose that $g : \mathbb{R} \to [0, \infty)$ is a bounded, piecewise continuous, deterministic function. Assume further that $g \in L^2([0, \infty))$ so that the Wiener integral

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s$$

is well defined for all $t \geq 0$. Define the continuous-time stochastic process $\{M_t, t \geq 0\}$ by setting

$$M_t = I_t^2 - \int_0^t g^2(s)\,\mathrm{d}s = \left(\int_0^t g(s)\,\mathrm{d}B_s\right)^2 - \int_0^t g^2(s)\,\mathrm{d}s.$$

Use Itô's formula to prove that $\{M_t, t \geq 0\}$ is a continuous-time martingale.

Solution. If

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s,$$

then $\mathrm{d}I_t = g(t)\,\mathrm{d}B_t$ so that $\mathrm{d}\langle I\rangle_t = g^2(t)\,\mathrm{d}t$. If

$$M_t = I_t^2 - \int_0^t g^2(s)\,\mathrm{d}s,$$

then written in differential form we have

$$\mathrm{d}M_t = \mathrm{d}(I_t^2) - g^2(t)\,\mathrm{d}t.$$

Version III of Itô's formula implies

$$\mathrm{d}(I_t^2) = 2I_t\,\mathrm{d}I_t + \mathrm{d}\langle I\rangle_t.$$

Substituting back therefore gives

$$\begin{aligned}\mathrm{d}M_t = \mathrm{d}(I_t^2) - g^2(t)\,\mathrm{d}t = 2I_t\,\mathrm{d}I_t + \mathrm{d}\langle I\rangle_t - g^2(t)\,\mathrm{d}t &= 2g(t)I_t\,\mathrm{d}B_t + g^2(t)\,\mathrm{d}t - g^2(t)\,\mathrm{d}t\\ &= 2g(t)I_t\,\mathrm{d}B_t.\end{aligned}$$

Since Itô integrals are martingales, we conclude that $\{M_t, t \ge 0\}$ is a continuous-time martingale.

Example 17.7 (Assignment #4, problem #9)**.** Suppose that $\{X_t, t \ge 0\}$ is a time-inhomogeneous Ornstein-Uhlenbeck-type process defined by the SDE

$$\mathrm{d}X_t = \sigma(t)\,\mathrm{d}B_t - a(X_t - g(t))\,\mathrm{d}t$$

where g and σ are (sufficiently regular) deterministic functions of time. If $Y_t = \exp\{X_t + ct\}$, use Itô's formula to compute $\mathrm{d}Y_t$.

Solution. If $\mathrm{d}X_t = \sigma(t)\,\mathrm{d}B_t - a(X_t - g(t))\,\mathrm{d}t$ and $Y_t = \exp\{X_t + ct\}$, then Version IV of Itô's formula implies that

$$\mathrm{d}Y_t = cY_t\,\mathrm{d}t + Y_t\,\mathrm{d}X_t + \frac{Y_t}{2}\,\mathrm{d}\langle X\rangle_t.$$

Since

$$\mathrm{d}\langle X\rangle_t = \sigma^2(t)dt,$$

we conclude that

$$\frac{\mathrm{d}Y_t}{Y_t} = \sigma(t)\,\mathrm{d}B_t + \left[c - a(X_t - g(t)) + \frac{\sigma^2(t)}{2}\right]\,\mathrm{d}t.$$

Since we want a stochastic differential equation for Y_t, we should really substitute back for X_t in terms of Y_t. Solving $Y_t = \exp\{X_t + ct\}$ for X_t gives $X_t = \log(Y_t) - ct$ so that

$$\begin{aligned}\frac{\mathrm{d}Y_t}{Y_t} &= \sigma(t)\,\mathrm{d}B_t + \left[c - a(\log(Y_t) - ct - g(t)) + \frac{\sigma^2(t)}{2}\right]\,\mathrm{d}t\\ &= \sigma(t)\,\mathrm{d}B_t + \left[c(1+at) - a\log(Y_t) + ag(t) + \frac{\sigma^2(t)}{2}\right]\,\mathrm{d}t.\end{aligned}$$

Example 17.8 (Assignment #4, problem #2)**.** Suppose that the price of a stock $\{X_t, t \geq 0\}$ follows geometric Brownian motion with drift 0.05 and volatility 0.3 so that it satisfies the stochastic differential equation

$$\mathrm{d}X_t = 0.3X_t \,\mathrm{d}B_t + 0.05X_t \,\mathrm{d}t.$$

If the price of the stock at time 2 is 30, determine the probability that the price of the stock at time 2.5 is between 30 and 33.

Solution. Since the price of the stock is given by geometric Brownian motion

$$\mathrm{d}X_t = 0.3X_t \,\mathrm{d}B_t + 0.05X_t \,\mathrm{d}t,$$

we can read off the solution, namely

$$X_t = X_0 \exp\left\{0.3B_t + \left(0.05 - \frac{0.3^2}{2}\right)t\right\} = X_0 \exp\{0.30B_t + 0.005t\}.$$

Therefore,

$$\begin{aligned}
&\mathbf{P}\{30 \leq X_{2.5} \leq 33 | X_2 = 30\} \\
&= \mathbf{P}\left\{\frac{\log\left(\frac{30}{X_0}\right) - 0.0125}{0.30} \leq B_{2.5} \leq \frac{\log\left(\frac{33}{X_0}\right) - 0.0125}{0.30} \,\middle|\, B_2 = \frac{\log\left(\frac{30}{X_0}\right) - 0.01}{0.30}\right\} \\
&= \mathbf{P}\left\{\frac{\log\left(\frac{30}{X_0}\right) - 0.0125}{0.30} - \frac{\log\left(\frac{30}{X_0}\right) - 0.01}{0.30} \leq B_{0.5} \leq \frac{\log\left(\frac{33}{X_0}\right) - 0.0125}{0.30} - \frac{\log\left(\frac{30}{X_0}\right) - 0.01}{0.30}\right\} \\
&= \mathbf{P}\left\{-\frac{0.0025}{0.30} \leq B_{0.5} \leq \frac{\log\left(\frac{33}{30}\right) - 0.0025}{0.30}\right\}
\end{aligned}$$

using the stationarity of Brownian increments. If $Z \sim \mathcal{N}(0,1)$ so that $B_{0.5} \sim \sqrt{0.5}\,Z$, then

$$\mathbf{P}\left\{-0.00833 \leq B_{0.5} \leq 0.3094\right\} = \mathbf{P}\{-0.0118 \leq Z \leq 0.4375\} = 0.1587.$$

Remark. The solution to the previous exercise can be generalized as follows. Suppose that $\{X_t, t \geq 0\}$ is geometric Brownian motion given by

$$\mathrm{d}X_t = \sigma X_t \,\mathrm{d}B_t + \mu X_t \,\mathrm{d}t$$

so that

$$X_t = X_0 \exp\left\{\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right\}.$$

If $s \geq 0$, $t > 0$, then

$$\log\left(\frac{X_{t+s}}{X_s}\right) = \sigma(B_{t+s} - B_s) + \left(\mu - \frac{\sigma^2}{2}\right)t.$$

Using the facts that (i) $B_{t+s} - B_s$ is independent of B_s, and (ii) $B_{t+s} - B_s \sim B_t \sim \mathcal{N}(0, t)$ implies that (i) $\log(X_{t+s}/X_s)$ is independent of $\log X_s$, and (ii)

$$\log\left(\frac{X_{t+s}}{X_s}\right) \sim \left(\frac{X_t}{X_0}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right) t\,,\, \sigma^2 t\right).$$

Therefore, we can conclude that if $0 < a < b$ and $c > 0$ are constants, then

$$\begin{aligned}
\mathbf{P}\{a \le X_{t+s} \le b | X_s = c\} &= \mathbf{P}\left\{\log\left(\frac{a}{c}\right) \le \log\left(\frac{X_{t+s}}{X_s}\right) \le \log\left(\frac{b}{c}\right)\right\} \\
&= \mathbf{P}\left\{\log\left(\frac{a}{c}\right) \le \log\left(\frac{X_t}{X_0}\right) \le \log\left(\frac{b}{c}\right)\right\} \\
&= \mathbf{P}\left\{\frac{\log\left(\frac{a}{c}\right) - \left(\mu - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}} \le Z \le \frac{\log\left(\frac{b}{c}\right) - \left(\mu - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}}\right\}
\end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$.

Example 17.9 (Assignment #4, problem #3)**.** Consider the Itô process $\{X_t, t \ge 0\}$ described by the stochastic differential equation

$$\mathrm{d}X_t = 0.10 X_t \,\mathrm{d}B_t + 0.25 X_t \,\mathrm{d}t.$$

Calculate the probability that X_t is at least 5% higher than X_0

(a) at time $t = 0.01$, and

(b) at time $t = 1$.

Solution. Since the price of the stock is given by geometric Brownian motion

$$\mathrm{d}X_t = 0.25 X_t \,\mathrm{d}t + 0.10 X_t \,\mathrm{d}B_t,$$

we can read off the solution, namely

$$X_t = X_0 \exp\left\{0.10 B_t + \left(0.25 - \frac{0.10^2}{2}\right) t\right\} = X_0 \exp\{0.10 B_t + 0.245 t\}.$$

Therefore, if $Z \sim \mathcal{N}(0, 1)$, then

$$\mathbf{P}\{X_t \ge 1.05 X_0\} = \mathbf{P}\left\{B_t \ge \frac{\log(1.05) - 0.245t}{0.10}\right\} = \mathbf{P}\left\{Z \ge \frac{\log(1.05) - 0.245t}{0.10\sqrt{t}}\right\}.$$

(a) If $t = 0.01$, then $\mathbf{P}\{X_{0.01} \ge 1.05 X_0\} = \mathbf{P}\{Z \ge 4.634\} = 0.000002$.

(b) If $t = 1$, then $\mathbf{P}\{X_1 \ge 1.05 X_0\} = \mathbf{P}\{Z \ge -1.962\} = 0.9751$.

Example 17.10 (Assignment #4, problem #4)**.** Consider the Itô process $\{X_t, t \geq 0\}$ described by the stochastic differential equation

$$\mathrm{d}X_t = 0.05 X_t \,\mathrm{d}B_t + 0.1 X_t \,\mathrm{d}t, \quad X_0 = 35.$$

Compute $\mathbf{P}\{X_5 \leq 48\}$.

Solution. Since the price of the stock is given by geometric Brownian motion

$$\mathrm{d}X_t = 0.1 X_t \,\mathrm{d}t + 0.05 X_t \,\mathrm{d}B_t, \quad X_0 = 35,$$

we can read off the solution, namely

$$X_t = 35 \exp\left\{0.05 B_t + \left(0.1 - \frac{0.05^2}{2}\right) t\right\} = 35 \exp\{0.05 B_t + 0.09875 t\}.$$

Therefore, if $Z \sim \mathcal{N}(0,1)$, then

$$\mathbf{P}\{X_5 \leq 48\} = \mathbf{P}\{B_5 \leq -3.5579\} = \mathbf{P}\left\{Z \leq \frac{-3.5579}{\sqrt{5}}\right\} = \mathbf{P}\{Z \leq -1.5911\} = 0.0558.$$

Example 17.11 (Assignment #4, problem #12)**.** It follows from Version II of Itô's formula that if $f(t,x)$ satisfies the partial differential equation

$$\dot{f}(t,x) + \frac{1}{2} f''(t,x) = 0,$$

then $f(t, B_t)$ is a martingale.

- If $f(t,x) = x^5 - 10tx^3 + 15t^2 x$, then $f(t, B_t)$ is a martingale.
- If $f(t,x) = x^6 - 15x^4 t + 45t^2 x^2 - 15t^3$, then $f(t, B_t)$ is a martingale.
- If $f(t,x) = e^{t/2} \cos(x)$, then $f(t, B_t)$ is a martingale.
- If $f(t,x) = -e^{-t/2} \cos(x)$, then $f(t, B_t)$ is a martingale.
- If $f(t,x) = e^{x - t/2}$, then $f(t, B_t)$ is a martingale.