# Lecture #26, 27: Solving the Black–Scholes Partial Differential Equation

Our goal for this lecture is to solve the Black-Scholes partial differential equation

$$\dot{V}(t,x) + \frac{\sigma^2}{2}x^2V''(t,x) + rxV'(t,x) - rV(t,x) = 0 \tag{16.1}$$

for $V(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}$, subject to the boundary condition

$$V(T,x) = (x-E)^+.$$

The first observation is that it suffices to solve (16.1) when $r = 0$. That is, if $W$ satisfies

$$\dot{W}(t,x) + \frac{\sigma^2}{2}x^2W''(t,x) = 0, \tag{16.2}$$

and $V(t,x) = e^{r(t-T)}W(t, e^{r(T-t)}x)$, then $V(t,x)$ satisfies (16.1) and $V(T,x) = W(T,x)$.

This can be checked by differentiation. There is, however, an "obvious" reason why it is true, namely due to the *time value of money* mentioned in Lecture #2. If money invested in a cash deposit grows at continuously compounded interest rate $r$, then $\$x$ at time $T$ is equivalent to $\$e^{r(t-T)}x$ at time $t$.

**Exercise 16.1.** Verify (using the multivariate chain rule) that if $W(t,x)$ satisfies (16.2) and $V(t,x) = e^{r(t-T)}W(t, e^{r(T-t)}x)$, then $V(t,x)$ satisfies (16.1) and $V(T,x) = W(T,x)$.

Since we have already seen that the Black-Scholes partial differential equation (16.1) does not depend on $\mu$, we can assume that $\mu = 0$. We have also just shown that it suffices to solve (16.1) when $r = 0$. Therefore, we will use $W$ to denote the Black-Scholes solution in the $r = 0$ case, i.e., the solution to (16.2), and we will then use $V$ as the solution in the $r > 0$ case, i.e., the solution to (16.1), where

$$V(t,x) = e^{r(t-T)}W(t, e^{r(T-t)}x). \tag{16.3}$$

We now note from (15.3) that the SDE for $W(t,S_t)$ is

$$\mathrm{d}W(t,S_t) = \sigma S_t W'(t,S_t)\,\mathrm{d}B_t + \left[\dot{W}(t,S_t) + \mu S_t W'(t,S_t) + \frac{\sigma^2}{2}S_t^2W''(t,S_t)\right]\mathrm{d}t.$$

We are assuming that $\mu = 0$ so that

$$\mathrm{d}W(t,S_t) = \sigma S_t W'(t,S_t)\,\mathrm{d}B_t + \left[\dot{W}(t,S_t) + \frac{\sigma^2}{2}S_t^2W''(t,S_t)\right]\mathrm{d}t.$$

We are also assuming that $W(t,x)$ satisfies the Black-Scholes PDE given by (16.2) which is exactly what is needed to make the $\mathrm{d}t$ term equal to 0. Thus, we have reduced the SDE for $W(t,S_t)$ to

$$\mathrm{d}W(t,S_t) = \sigma S_t W'(t,S_t)\,\mathrm{d}B_t.$$

We now have a stochastic differential equation with no $\mathrm{d}t$ term which means, using Theorem 12.6, that $W(t,S_t)$ is a martingale. Formally, if $M_t = W(t,S_t)$, then the stochastic process $\{M_t, t \geq 0\}$ is a martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \geq 0\}$.

Next, we use the fact that martingales have stable expectation at fixed times to conclude that

$$\mathbb{E}(M_0) = \mathbb{E}(M_T).$$

Since we know the value of the European call option at time $T$ is $W(T,S_T) = (S_T - E)^+$, we see that

$$M_T = W(T,S_T) = (S_T - E)^+.$$

Furthermore, $M_0 = W(0,S_0)$ is non-random (since $S_0$, the stock price at time 0, is known), and so we conclude that $M_0 = \mathbb{E}(M_T)$ which implies

$$W(0,S_0) = \mathbb{E}[\,(S_T - E)^+\,]. \tag{16.4}$$

The final step is to actually calculate the expected value in (16.4). Since we are assuming $\mu = 0$, the stock price follows geometric Brownian motion $\{S_t, t \geq 0\}$ where

$$S_t = S_0 \exp\left\{\sigma B_t - \frac{\sigma^2}{2} t\right\}.$$

Hence, at time $T$, we need to consider the random variable

$$S_T = S_0 \exp\left\{\sigma B_T - \frac{\sigma^2}{2} T\right\}.$$

We know $B_T \sim \mathcal{N}(0,T)$ so that we can write

$$S_T = S_0\, e^{-\frac{\sigma^2 T}{2}}\, e^{\sigma\sqrt{T} Z}$$

for $Z \sim \mathcal{N}(0,1)$. Thus, we can now use the result of Exercise 3.7, namely if $a > 0$, $b > 0$, $c > 0$ are constants and $Z \sim \mathcal{N}(0,1)$, then

$$\mathbb{E}[\,(ae^{bZ} - c)^+\,] = ae^{b^2/2}\,\Phi\left(b + \frac{1}{b}\log\frac{a}{c}\right) - c\,\Phi\left(\frac{1}{b}\log\frac{a}{c}\right), \tag{16.5}$$

with

$$a = S_0\, e^{-\frac{\sigma^2 T}{2}}, \quad b = \sigma\sqrt{T}, \quad c = E$$

to conclude

$$\begin{aligned}
&\mathbb{E}[\,(S_T - E)^+\,] \\
&\quad = S_0\, e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}}\, \Phi\left(\sigma\sqrt{T} + \frac{1}{\sigma\sqrt{T}} \log \frac{S_0\, e^{-\frac{\sigma^2 T}{2}}}{E}\right) - E\,\Phi\left(\frac{1}{\sigma\sqrt{T}} \log \frac{S_0\, e^{-\frac{\sigma^2 T}{2}}}{E}\right) \\
&\quad = S_0\,\Phi\left(\frac{1}{\sigma\sqrt{T}} \log \frac{S_0}{E} + \frac{\sigma\sqrt{T}}{2}\right) - E\,\Phi\left(\frac{1}{\sigma\sqrt{T}} \log \frac{S_0}{E} - \frac{\sigma\sqrt{T}}{2}\right).
\end{aligned}$$

To account for the *time value of money*, we can use Exercise 16.1 to give the solution for $r > 0$. That is, if $V(0, S_0)$ denotes the fair price (at time 0) of a European call option with strike price $E$, then using (16.3) we conclude

$$\begin{aligned}
V(0,S_0) &= e^{-rT} W(0, e^{rT} S_0) \\
&= e^{-rT} e^{rT} S_0\,\Phi\left(\frac{1}{\sigma\sqrt{T}} \log \frac{e^{rT} S_0}{E} + \frac{\sigma\sqrt{T}}{2}\right) - E e^{-rT}\,\Phi\left(\frac{1}{\sigma\sqrt{T}} \log \frac{e^{rT} S_0}{E} - \frac{\sigma\sqrt{T}}{2}\right) \\
&= S_0\,\Phi\left(\frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - E e^{-rT}\,\Phi\left(\frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= S_0\,\Phi\,(d_1) - E e^{-rT}\,\Phi\,(d_2)
\end{aligned}$$

where

$$d_1 = \frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log(S_0/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

AWESOME!

**Remark.** We have now arrived at equation (8.19) on page 80 of Higham [11]. Note that Higham only *states* the answer; he never actually goes through the solution of the Black-Scholes PDE.

**Summary.** Let's summarize what we did. We assumed that the asset $S$ followed geometric Brownian motion given by

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t,$$

and that the risk-free bond $D$ grew at continuously compounded interest rate $r$ so that

$$\mathrm{d}D(t, S_t) = rD(t, S_t)\,\mathrm{d}t.$$

Using Version IV of Itô's formula on the value of the option $V(t, S_t)$ combined with the self-financing portfolio implied by the no arbitrage assumption led to the Black-Scholes partial differential equation

$$\dot{V}(t,x) + \frac{\sigma^2}{2} x^2 V''(t,x) + rxV'(t,x) - rV(t,x) = 0.$$

We also made the important observation that this PDE does not depend on $\mu$. We then saw that it was sufficient to consider $r = 0$ since we noted that if $W(t, x)$ solved the resulting PDE

$$\dot{W}(t,x) + \frac{\sigma^2}{2}x^2 W''(t,x) = 0,$$

then $V(t, x) = e^{r(t-T)}W(t, e^{r(T-t)}x)$ solved the Black-Scholes PDE for $r > 0$. We then assumed that $\mu = 0$ and we found the SDE for $W(t, S_t)$ which had only a $\mathrm{d}B_t$ term (and no $\mathrm{d}t$ term). Using the fact that Itô integrals are martingales implied that $\{W(t, S_t), t \geq 0\}$ was a martingale, and so the stable expectation property of martingales led to the equation

$$W(0, S_0) = \mathbb{E}(W(T, S_T)).$$

Since we knew that $V(T, S_T) = W(T, S_T) = (S_T - E)^+$ for a European call option, we could compute the resulting expectation. We then translated back to the $r > 0$ case via

$$V(0, S_0) = e^{-rT}W(0, e^{rT}S_0).$$

This previous observation is extremely important since it tells us precisely how to price European call options with different payoffs. In general, if the payoff function at time $T$ is $\Lambda(x)$ so that

$$V(T, x) = W(T, x) = \Lambda(x),$$

then, since $\{W(t, S_t), t \geq 0\}$ is a martingale,

$$W(0, S_0) = \mathbb{E}(W(T, S_T)) = \mathbb{E}(\Lambda(S_T)).$$

By assuming that $\mu = 0$, we can write $S_T$ as

$$S_T = S_0 \exp\left\{\sigma B_T - \frac{\sigma^2}{2}T\right\} = S_0\, e^{-\frac{\sigma^2 T}{2}}\, e^{\sigma\sqrt{T}Z}$$

with $Z \sim \mathcal{N}(0, 1)$. Therefore, if $\Lambda$ is sufficiently nice, then $\mathbb{E}(\Lambda(S_T))$ can be calculated explicitly, and we can use

$$V(0, S_0) = e^{-rT}W(0, e^{rT}S_0)$$

to determine the required fair price to pay at time 0.

In particular, we can follow this strategy to answer the following question posed at the end of Lecture #1.

**Example 16.2.** In the Black-Scholes world, price a European option with a payoff of $\max\{S_T^2 - K, 0\}$ at time $T$.

**Solution.** The required time 0 price is $V(0, S_0) = e^{-rT}W(0, e^{rT}S_0)$ where $W(0, S_0) = \mathbb{E}[\,(S_T^2 - K)^+\,]$. Since we can write

$$S_T^2 = S_0^2\, e^{-\sigma^2 T}\, e^{2\sigma\sqrt{T}Z}$$

with $Z \sim \mathcal{N}(0, 1)$, we can use (16.5) with

$$a = S_0^2\, e^{-\sigma^2 T}, \quad b = 2\sigma\sqrt{T}, \quad c = K$$

to conclude

$$V(0, S_0) = S_0^2\, e^{(\sigma^2+r)T}\, \Phi\left(\frac{\log(S_0^2/K) + (2r + 3\sigma^2)T}{2\sigma\sqrt{T}}\right) - Ke^{-rT}\, \Phi\left(\frac{\log(S_0^2/K) + (2r - \sigma^2)T}{2\sigma\sqrt{T}}\right).$$