Lecture #20, 21, 22: Itô's Formula (Part II)

Recall from last lecture that we derived Itô's formula, namely if $f(x) \in C^2(\mathbb{R})$, then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s. \tag{14.1}$$

The derivation of Itô's formula involved carefully manipulating Taylor's theorem for the function $f(x)$. (In fact, the actual proof of Itô's formula follows a careful analysis of Taylor's theorem for a function of one variable.) As you may know from MATH 213, there is a version of Taylor's theorem for functions of two variables. Thus, by writing down Taylor's theorem for the function $f(t,x)$ and carefully checking which higher order terms disappear, one can derive the following generalized version of Itô's formula.

Consider those functions of two variables, say $f(t,x)$, which have one continuous derivative in the "t-variable" for $t \ge 0$, and two continuous derivatives in the "x-variable." If f is such a function, we say that $f \in C^1([0,\infty)) \times C^2(\mathbb{R})$.

Theorem 14.1 (Generalized Version of Itô's Formula)**.** *If $f \in C^1([0,\infty)) \times C^2(\mathbb{R})$, then*

$$f(t,B_t) - f(0,B_0) = \int_0^t \frac{\partial}{\partial x} f(s,B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s,B_s)\,\mathrm{d}s + \int_0^t \frac{\partial}{\partial t} f(s,B_s)\,\mathrm{d}s. \tag{14.2}$$

Remark. It is traditional to use the variables t and x for the function $f(t,x)$ of two variables in the generalized version of Itô's formula. This has the unfortunate consequence that the letter t serves both as a dummy variable for the function $f(t,x)$ and as a time variable in the upper limit of integration. One way around this confusion is to use the prime ($'$) notation for derivatives in the space variable (the x-variable) and the dot ($\dot{}$) notation for derivatives in the time variable (the t-variable). That is,

$$f'(t,x) = \frac{\partial}{\partial x} f(t,x), \quad f''(t,x) = \frac{\partial^2}{\partial x^2} f(t,x), \quad \dot{f}(t,x) = \frac{\partial}{\partial t} f(t,x),$$

and so (14.2) becomes

$$f(t,B_t) - f(0,B_0) = \int_0^t f'(s,B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(s,B_s)\,\mathrm{d}s + \int_0^t \dot{f}(s,B_s)\,\mathrm{d}s.$$

Example 14.2. Let $f(t,x) = tx^2$ so that

$$f'(t,x) = 2xt, \quad f''(t,x) = 2t, \quad \text{and} \quad \dot{f}(t,x) = x^2.$$

Therefore, the generalized version of Itô's formula implies

$$tB_t^2 = \int_0^t 2sB_s\,\mathrm{d}B_s + \frac{1}{2}\int_0^t 2s\,\mathrm{d}s + \int_0^t B_s^2\,\mathrm{d}s.$$

Upon rearranging we conclude

$$\int_0^t sB_s\,\mathrm{d}B_s = \frac{1}{2}\left(tB_t^2 - \frac{t^2}{2} - \int_0^t B_s^2\,\mathrm{d}s\right).$$

Example 14.3. Let $f(t,x) = \frac{1}{3}x^3 - xt$ so that

$$f'(t,x) = x^2 - t, \quad f''(t,x) = 2x, \quad \text{and} \quad \dot{f}(t,x) = -x.$$

Therefore, the generalized version of Itô's formula implies

$$\frac{1}{3}B_t^3 - tB_t = \int_0^t (B_s^2 - s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t 2B_s\,\mathrm{d}s - \int_0^t B_s\,\mathrm{d}s = \int_0^t (B_s^2 - s)\,\mathrm{d}B_s$$

which gives the same result as was obtained in Example 13.4.

Example 14.4. If we combine our result of Example 13.5, namely

$$\int_0^t B_s^3\,\mathrm{d}B_s = \frac{1}{4}B_t^4 - \frac{3}{2}\int_0^t B_s^2\,\mathrm{d}s,$$

with our result of Example 14.2, namely

$$\int_0^t sB_s\,\mathrm{d}B_s = \frac{1}{2}\left(tB_t^2 - \frac{t^2}{2} - \int_0^t B_s^2\,\mathrm{d}s\right),$$

then we conclude that

$$\int_0^t (B_s^3 - 3sB_s)\,\mathrm{d}B_s = \frac{1}{4}B_t^4 - \frac{3}{2}tB_t^2 + \frac{3}{4}t^2.$$

Example 14.5. If we re-write the results of Example 13.4 and Example 14.4 slightly differently, then we see that

$$\int_0^t 3(B_s^2 - s)\,\mathrm{d}B_s = B_t^3 - 3tB_t$$

and

$$\int_0^t 4(B_s^3 - 3sB_s)\,\mathrm{d}B_s = B_t^4 - 6tB_t^2 + 3t^2.$$

The reason for doing this is that Theorem 12.6 tells us that Itô integrals are martingales. Hence, we see that $\{B_t^3 - 3tB_t, t \geq 0\}$ and $\{B_t^4 - 6tB_t^2 + 3t^2, t \geq 0\}$ must therefore be martingales with respect to the Brownian filtration $\{\mathcal{F}_t, t \geq 0\}$. Look back at Exercise 5.6; you have already verified that these are martingales. Of course, using Itô's formula makes for a much easier proof.

Exercise 14.6. Prove that the process $\{M_t, t \geq 0\}$ defined by setting

$$M_t = \exp\left\{\theta B_t - \frac{\theta^2 t}{2}\right\}$$

where $\theta \in \mathbb{R}$ is a constant is a martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \geq 0\}$.

Example 14.7. We will now show that Theorem 9.2, the integration-by-parts formula for Wiener integrals, is a special case of the generalized version of Itô's formula. Suppose that $g : [0,\infty) \to \mathbb{R}$ is a bounded, continuous function in $L^2([0,\infty))$. Suppose further that g is differentiable with g' also bounded and continuous. Let $f(t,x) = xg(t)$ so that

$$f'(t,x) = g(t), \quad f''(t,x) = 0, \quad \text{and} \quad \dot{f}(t,x) = xg'(t).$$

Therefore, the generalized version of Itô's formula implies

$$g(t)B_t = \int_0^t g(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t 0\,\mathrm{d}s + \int_0^t g'(s)B_s\,\mathrm{d}s.$$

Rearranging gives

$$\int_0^t g(s)\,\mathrm{d}B_s = g(t)B_t - \int_0^t g'(s)B_s\,\mathrm{d}s$$

as required.

There are a number of versions of Itô's formula that we will use; the first two we have already seen. The easiest way to remember all of the different versions is as a *stochastic differential equation* (or SDE).

Theorem 14.8 (Version I). *If $f \in C^2(\mathbb{R})$, then*

$$\mathrm{d}f(B_t) = f'(B_t)\,\mathrm{d}B_t + \frac{1}{2}f''(B_t)\,\mathrm{d}t.$$

Theorem 14.9 (Version II). *If $f \in C^1([0,\infty)) \times C^2(\mathbb{R})$, then*

$$\begin{aligned}\mathrm{d}f(t,B_t) &= f'(t,B_t)\,\mathrm{d}B_t + \frac{1}{2}f''(t,B_t)\,\mathrm{d}t + \dot{f}(t,B_t)\,\mathrm{d}t\\ &= f'(t,B_t)\,\mathrm{d}B_t + \left[\dot{f}(t,B_t) + \frac{1}{2}f''(t,B_t)\right]\,\mathrm{d}t.\end{aligned}$$

Example 14.10. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion. Determine the SDE satisfied by

$$X_t = \exp\{\sigma B_t + \mu t\}.$$

Solution. Consider the function $f(t,x) = \exp\{\sigma x + \mu t\}$. Since

$$f'(t,x) = \sigma\exp\{\sigma x + \mu t\}, \quad f''(t,x) = \sigma^2\exp\{\sigma x + \mu t\}, \quad \dot{f}(t,x) = \mu\exp\{\sigma x + \mu t\},$$

it follows from Version II of Itô's formula that

$$\mathrm{d}f(t,B_t) = \sigma\exp\{\sigma B_t + \mu t\}\,\mathrm{d}B_t + \frac{\sigma^2}{2}\exp\{\sigma B_t + \mu t\}\,\mathrm{d}t + \mu\exp\{\sigma B_t + \mu t\}\,\mathrm{d}t.$$

In other words,

$$\mathrm{d}X_t = \sigma X_t\,\mathrm{d}B_t + \left(\frac{\sigma^2}{2} + \mu\right)X_t\,\mathrm{d}t.$$

Suppose that the stochastic process $\{X_t, t \geq 0\}$ is defined by the stochastic differential equation

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t$$

where a and b are *suitably smooth* functions. We call such a stochastic process a *diffusion* (or an *Itô diffusion* or an *Itô process*).

Again, a careful analysis of Taylor's theorem provides a version of Itô's formula for diffusions.

Theorem 14.11 (Version III)**.** *Let X_t be a diffusion defined by the SDE*

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t.$$

If $f \in C^2(\mathbb{R})$, then

$$\mathrm{d}f(X_t) = f'(X_t)\,\mathrm{d}X_t + \frac{1}{2}f''(X_t)\,\mathrm{d}\langle X\rangle_t$$

where $\mathrm{d}\langle X\rangle_t$ is computed as

$$\mathrm{d}\langle X\rangle_t = (\mathrm{d}X_t)^2 = [a(t, X_t)\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t]^2 = a^2(t, X_t)\,\mathrm{d}t$$

using the rules $(\mathrm{d}B_t)^2 = \mathrm{d}t$, $(\mathrm{d}t)^2 = 0$, $(\mathrm{d}B_t)(\mathrm{d}t) = (\mathrm{d}t)(\mathrm{d}B_t) = 0$. That is,

$$\begin{aligned}
\mathrm{d}f(X_t) &= f'(X_t)\,[a(t, X_t)\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t] + \frac{1}{2}f''(X_t)a^2(t, X_t)\,\mathrm{d}t \\
&= f'(X_t)a(t, X_t)\,\mathrm{d}B_t + f'(X_t)b(t, X_t)\,\mathrm{d}t + \frac{1}{2}f''(X_t)a^2(t, X_t)\,\mathrm{d}t \\
&= f'(X_t)a(t, X_t)\,\mathrm{d}B_t + \left[f'(X_t)b(t, X_t) + \frac{1}{2}f''(X_t)a^2(t, X_t)\right]\,\mathrm{d}t.
\end{aligned}$$

And finally we give the version of Itô's formula for diffusions for functions $f(t, x)$ of two variables.

Theorem 14.12 (Version IV)**.** *Let X_t be a diffusion defined by the SDE*

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t.$$

If $f \in C^1([0, \infty)) \times C^2(\mathbb{R})$, then

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= f'(t, X_t)\,\mathrm{d}X_t + \frac{1}{2}f''(t, X_t)\,\mathrm{d}\langle X\rangle_t + \dot{f}(t, X_t)\,\mathrm{d}t \\
&= f'(t, X_t)a(t, X_t)\,\mathrm{d}B_t + \left[\dot{f}(t, X_t) + f'(t, X_t)b(t, X_t) + \frac{1}{2}f''(t, X_t)a^2(t, X_t)\right]\,\mathrm{d}t
\end{aligned}$$

again computing $\mathrm{d}\langle X\rangle_t = (\mathrm{d}X_t)^2$ using the rules $(\mathrm{d}B_t)^2 = \mathrm{d}t$, $(\mathrm{d}t)^2 = 0$, $(\mathrm{d}B_t)(\mathrm{d}t) = (\mathrm{d}t)(\mathrm{d}B_t) = 0$.