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Lecture #2: Financial Option Valuation Preliminaries

Recall that a *portfolio* describes a combination of

(i) assets (i.e., stocks),

(ii) options, and

(iii) cash invested in a bank, i.e., bonds.

We will write $S(t)$ to denote the value of an asset at time $t \geq 0$. Since an asset is defined as a financial object whose value is known at present but is liable to change over time, we see that it is reasonable to *model* the asset price (i.e., stock price) by a stochastic process $\{S_t, t \geq 0\}$. There will be much to say about this later.

Suppose that $D(t)$ denotes the value at time t of an investment which grows according to a continuously compounded interest rate r. That is, suppose that an amount D_0 is invested at time 0. Its value at time $t \geq 0$ is given by

$$D(t) = e^{rt} D_0. \tag{2.1}$$

There are a couple of different ways to derive this formula for compound interest. One way familiar to actuarial science students is as the solution of a *constant force of interest equation.* That is, $D(t)$ is the solution of the equation

$$\delta_t = r \text{ with } r > 0$$

where

$$\delta_t = \frac{\mathrm{d}}{\mathrm{d}t} \log D(t)$$

and initial condition $D(0) = D_0$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t} \log D(t) = r \quad \text{implies} \quad \frac{D'(t)}{D(t)} = r$$

so that $D'(t) = rD(t)$. This differential equation can then be solved by separation-of-variables giving (2.1).

Remark. We will use $D(t)$ as our model of the risk-free savings account, or bond. Assuming that such a bond exists means that having \$1 at time 0 or \$$e^{rt}$ at time t are both of equal value. Equivalently, having \$1 at time t or \$$e^{-rt}$ at time 0 are both of equal value. This is sometimes known as the *time value of money.* Transferring money in this way is known as *discounting for interest* or *discounting for inflation.*

The word *arbitrage* is a fancy way of saying "money for nothing." One of the fundamental assumptions that we will make is that of *no arbitrage* (informally, we might call this the *no free lunch* assumption).

The form of the *no arbitrage* assumption given in Higham [11] is as follows.

> There is never an opportunity to make a risk-free profit that gives a greater return than that provided by interest from a bank deposit.

Note that this only applies to *risk-free* profit.

Example 2.1. Suppose that a company has offices in Toronto and London. The exchange rate between the dollar and the pound must be the same in both cities. If the exchange rate were $\$1.80 = £1$ in Toronto but only $\$1.78 = £1$ in London, then the company could instantly sell pounds in Toronto for \$1.80 each and buy them back in London for only \$1.78 making a risk-free profit of \$0.02 per pound. This would lead to unlimited profit for the company. Others would then execute the same trades leading to more unlimited profit and a total collapse of the market! Of course, the market would never allow such an *obvious discrepancy* to exist for any period of time.

The scenario described in the previous example is an illustration of an economic law known as *the law of one price* which states that "in an *efficient* market all identical goods must have only one price." An obvious violation of the efficient market assumption is found in the pricing of gasoline. Even in Regina, one can often find two gas stations on opposite sides of the street selling gas at different prices! (Figuring out how to legally take advantage of such a discrepancy is another matter altogether!)

The job of *arbitrageurs* is to scour the markets looking for arbitrage opportunities in order to make risk-free profit. The website

`http://www.arbitrageview.com/riskarb.htm`

lists some arbitrage opportunities in pending merger deals in the U.S. market. The following quote from this website is also worth including.

> "It is important to note that merger arbitrage is not a complete risk free strategy. Profiting on the discount spread may look like the closest thing to a free lunch on Wall Street, however there are number of risks such as the probability of a deal failing, shareholders voting down a deal, revising the terms of the merger, potential lawsuits, etc. In addition the trading discount captures the time value of money for the period between the announcement and the closing of the deal. Again the arbitrageurs face the risk of a deal being prolonged and achieving smaller rate of return on an annualized basis."

Nonetheless, in order to derive a reasonable mathematical model of a financial market we must not allow for arbitrage opportunities.

A neat little argument gives the relationship between the value (at time 0) of a European call option C and the value (at time 0) of a European put option P (with both options being on the same asset S at the same expiry date T and same strike price E). This is known as the so-called *put-call parity for European options.*

Consider two portfolios Π_1 and Π_2 where (at time 0)

- Π_1 consists of one call option plus Ee^{-rT} invested in a risk-free bond, and
- Π_2 consists of one put option plus one unit of the asset $S(0)$.

At the expiry date T, the portfolio Π_1 is worth $\max\{S(T) - E, 0\} + E = \max\{S(T), E\}$, and the portfolio Π_2 is worth $\max\{E - S(T), 0\} + S(T) = \max\{S(T), E\}$. Hence, since both portfolios always give the same payoff, the no arbitrage assumption (or simply common sense) dictates that they have the same value at time 0. Thus,

$$C + Ee^{-rT} = P + S(0). \tag{2.2}$$

It is important to note that we have not figured out a fair value at time 0 for a European call option (or a European put option). We have only concluded that it is sufficient to price the European call option, because the value of the European put option follows immediately from (2.2). We will return to this result when we solve the Black-Scholes partial differential equation.

Summary. We assume that it is possible to hold a portfolio of stocks and bonds. Both can be freely traded, and we can hold negative amounts of each without penalty. (That is, we can short-sell either instrument at no cost.) The stock is a risky asset which can be bought or sold (or even short-sold) in arbitrary units. Furthermore, it does not pay dividends. The bond, on the other hand, is a risk-free investment. The money invested in a bond is secure and grows according to a continuously compounded interest rate r. Trading takes place in continuous time, there are no transaction costs, and we will not be concerned with the bid-ask spread when pricing options. We trade in an efficient market in which arbitrage opportunities do not exist.

Example 2.2 (Pricing a forward contract)**.** As already noted, our primary goal is to determine the fair price to pay (at time 0) for a European call option. The call option is only one example of a financial derivative. The oldest derivative, and arguably the most natural claim on a stock, is the *forward.*

If two parties enter into a *forward contract* (at time 0), then one party (the seller) agrees to give the other party (the holder) the specified stock at some prescribed time in the future for some prescribed price.

Suppose that T denotes the expiry date, F denotes the strike price, and the value of the stock at time $t > 0$ is $S(t)$.

Note that a forward is not the same as a European call option. The stock *must* change hands at time T for \$$F$. The contract dictates that the seller is obliged to produce the stock at time T and that the holder is obliged to pay \$$F$ for the stock. Thus, the time T value of the forward contract for the holder is $S(T) - F$, and the time T value for the seller is $F - S(T)$.

Since money will change hands at time T, to determine the *fair value* of this contract means to determine the value of F.

Suppose that the distribution of the stock at time T is known. That is, suppose that $S(T)$ is a random variable having a known continuous distribution with density function f. The expected value of $S(T)$ is therefore

$$\mathbb{E}[S(T)] = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x.$$

Thus, the expected value at time T of the forward contract is

$$\mathbb{E}[S(T) - F]$$

(which is *calculable exactly* since the distribution of $S(T)$ is known). This suggests that the fair value of the strike price should satisfy

$$0 = \mathbb{E}[S(T) - F] \text{ so that } F = \mathbb{E}[S(T)].$$

In fact, the strong law of large numbers *justifies* this calculation—in the long run, the average of outcomes tends towards the expected value of a single outcome. In other words, the law of large numbers *suggests* that the fair strike price is $F = \mathbb{E}[S(T)]$.

The problem is that this price is not enforceable. That is, although our calculation is not incorrect, it does lead to an arbitrage opportunity. Thus, in order to show that expectation pricing is not enforceable, we need to construct a portfolio which allows for an arbitrage opportunity.

Consider the seller of the contract obliged to deliver the stock at time T in exchange for $\$F$. The seller borrows $S(0)$ now, buys the stock, puts it in a drawer, and waits. At time T, the seller then repays the loan for $e^{rT}S(0)$ but has the stock ready to deliver. Thus, if the strike price is less that $e^{rT}S(0)$, the seller will lose money with certainty. If the strike price is more than $e^{rT}S(0)$, the seller will make money with certainty.

Of course, the holder of the contract can run this scheme in reverse. Thus, writing more than $e^{rT}S(0)$ will mean that the holder will lose money with certainty.

Hence, the only *fair value* for the strike price is $F = e^{rT}S(0)$.

Remark. To put it quite simply, if there is an arbitrage price, then any other price is too dangerous to quote. Notice that the no arbitrage price for the forward contract completely ignores the randomness in the stock. If $\mathbb{E}[S(T)] > F$, then the holder of a forward contract *expects* to make money. However, so do holders of the stock itself!

Remark. Both a forward contract and a *futures contract* are contracts whereby the seller is obliged to deliver the prescribed asset to the holder at the prescribed time for the prescribed price. There are, however, two main differences. The first is that futures are traded on an exchange, while forwards are traded over-the-counter. The second is that futures are margined, while forwards are not. These matters will not concern us in this course.