Statistics 441 (Fall 2014) October 8, 2014
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Lecture #15: Itô Integration (Part I)

Recall that for bounded, piecewise continuous deterministic $L^2([0,\infty))$ functions, we have defined the Wiener integral

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s$$

which satisfied the following properties:

- $I_0 = 0$,
- for fixed $t > 0$, the random variable I_t is normally distributed with mean 0 and variance

$$\int_0^t g^2(s)\,\mathrm{d}s,$$

- the stochastic process $\{I_t,\ t \ge 0\}$ is a martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \ge 0\}$, and
- the trajectory $t \mapsto I_t$ is continuous.

Our goal for the next two lectures is to define the integral

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s. \tag{11.1}$$

for *random* functions g.

We understand from our work on Wiener integrals that for fixed $t > 0$ the stochastic integral I_t must be a random variable depending on the Brownian sample path. Thus, the interpretation of (11.1) is as follows. Fix a realization (or sample path) of Brownian motion $\{B_t(\omega), t \ge 0\}$ and a realization (depending on the Brownian sample path observed) of the stochastic process $\{g(t,\omega), t \ge 0\}$ so that, for fixed $t > 0$, the integral (11.1) is really a random variable, namely

$$I_t(\omega) = \int_0^t g(s,\omega)\,\mathrm{d}B_s(\omega).$$

We begin with the example where g is a Brownian motion. This seemingly simple example will serve to illustrate more of the subtleties of integration with respect to Brownian motion.

Example 11.1. Suppose that $\{B_t,\ t \ge 0\}$ is a Brownian motion with $B_0 = 0$. We would like to compute

$$I_t = \int_0^t B_s\,\mathrm{d}B_s$$

for this particular realization $\{B_t, t \geq 0\}$ of Brownian motion. If Riemann integration were valid, we would expect, using the fundamental theorem of calculus, that

$$I_t = \int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}(B_t^2 - B_0^2) = \frac{1}{2}B_t^2. \tag{11.2}$$

Motivated by our experience with Wiener integration, we expect that I_t has mean 0. However, if I_t is given by (11.2), then

$$\mathbb{E}(I_t) = \frac{1}{2}\mathbb{E}(B_t^2) = \frac{t}{2}.$$

We might also expect that the stochastic process $\{I_t, \; t \geq 0\}$ is a martingale; of course, $\{B_t^2/2, \; t \geq 0\}$ is *not* a martingale, although,

$$\left\{ \frac{1}{2}B_t^2 - \frac{t}{2}, \; t \geq 0 \right\} \tag{11.3}$$

is a martingale. Is it possible that the value of I_t is given by (11.3) instead? We will now show that yes, in fact,

$$\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{t}{2}.$$

Suppose that $\pi_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = t\}$ is a partition of $[0, t]$ and let

$$L_n = \sum_{i=1}^n B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) \quad \text{and} \quad R_n = \sum_{i=1}^n B_{t_i}(B_{t_i} - B_{t_{i-1}})$$

denote the left-hand and right-hand Riemann sums, respectively. Observe that

$$R_n - L_n = \sum_{i=1}^n B_{t_i}(B_{t_i} - B_{t_{i-1}}) - \sum_{i=1}^n B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2. \tag{11.4}$$

The next theorem shows that

$$(R_n - L_n) \not\to 0 \;\; \text{as} \;\; \operatorname{mesh}(\pi_n) = \max_{i \leq i \leq n}(t_i - t_{i-1}) \to 0$$

which implies that the attempted Riemann integration (11.2) is not valid for Brownian motion.

Theorem 11.2. *If* $\{\pi_n, \; n = 1, 2, 3, \ldots\}$ *is a refinement of* $[0, t]$ *with* $\operatorname{mesh}(\pi_n) \to 0$*, then*

$$\sum_{i=1}^n \left(B_{t_i} - B_{t_{i-1}}\right)^2 \to t \quad \textit{in } L^2$$

as $\operatorname{mesh}(\pi_n) \to 0$.

Proof. To begin, notice that

$$\sum_{i=1}^{n}(t_i - t_{i-1}) = t.$$

Let

$$Y_n = \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2 - t = \sum_{i=1}^{n} \left[\left(B_{t_i} - B_{t_{i-1}}\right)^2 - (t_i - t_{i-1})\right] = \sum_{i=1}^{n} X_i$$

where

$$X_i = \left(B_{t_i} - B_{t_{i-1}}\right)^2 - (t_i - t_{i-1}),$$

and note that

$$Y_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i\, X_j = \sum_{i=1}^{n} X_i^2 + 2\sum_{i<j} X_i\, X_j.$$

The independence of the Brownian increments implies that $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$; hence,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2).$$

But

$$\begin{aligned}\mathbb{E}(X_i^2) &= \mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})^4\right] - 2(t_i - t_{i-1})\mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})^2\right] + (t_i - t_{i-1})^2 \\ &= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\ &= 2(t_i - t_{i-1})^2\end{aligned}$$

since the fourth moment of a normal random variable with mean 0 and variance $t_i - t_{i-1}$ is $3(t_i - t_{i-1})^2$. Therefore,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) = 2\sum_{i=1}^{n}(t_i - t_{i-1})^2 \le 2 \text{ mesh}(\pi_n) \sum_{i=1}^{n}(t_i - t_{i-1}) = 2t \text{ mesh}(\pi_n) \to 0$$

as $\text{mesh}(\pi_n) \to 0$ from which we conclude that $\mathbb{E}(Y_n^2) \to 0$ as $\text{mesh}(\pi_n) \to 0$. However, this is exactly what it means for $Y_n \to 0$ in L^2 as $\text{mesh}(\pi_n) \to 0$, and the proof is complete. □

As a result of this theorem, we define the *quadratic variation* of Brownian motion to be this limit in L^2.

Definition 11.3. The *quadratic variation* of a Brownian motion $\{B_t, t \ge 0\}$ on the interval $[0, t]$ is defined to be

$$Q_2(B[0,t]) = t \quad (\text{in } L^2).$$

Since

$$(R_n - L_n) \to t \text{ in } L^2 \text{ as } \text{mesh}(\pi_n) \to 0$$

we see that L_n and R_n cannot possibly have the same limits in L^2. This is not necessarily surprising since $B_{t_{i-1}}$ is independent of $B_{t_i} - B_{t_{i-1}}$ from which it follows that $\mathbb{E}(L_n) = 0$ while $\mathbb{E}(R_n) = t$.

Exercise 11.4. Show that $\mathbb{E}(L_n) = 0$ and $\mathbb{E}(R_n) = t$.

On the other hand,

$$\begin{aligned} R_n + L_n = \sum_{i=1}^{n} B_{t_i}(B_{t_i} - B_{t_{i-1}}) + \sum_{i=1}^{n} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) &= \sum_{i=1}^{n}(B_{t_i} + B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \\ &= \sum_{i=1}^{n}(B_{t_i}^2 - B_{t_{i-1}}^2) \\ &= B_{t_n}^2 - B_{t_0}^2 \\ &= B_t^2 - B_0^2 \\ &= B_t^2. \end{aligned} \tag{11.5}$$

Thus, from (11.4) and (11.5) we conclude that

$$L_n = \frac{1}{2}\left(B_t^2 - \sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right) \quad \text{and} \quad R_n = \frac{1}{2}\left(B_t^2 + \sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right)$$

and so

$$L_n \to \frac{1}{2}(B_t^2 - t) \quad \text{in } L^2 \quad \text{and} \quad R_n \to \frac{1}{2}(B_t^2 + t) \quad \text{in } L^2.$$

Unlike the usual Riemann integral, the limit of these sums *does* depend on the intermediate points used (i.e., left– or right-endpoints). However, $\{B_t^2 + t,\ t \geq 0\}$ is not a martingale, although $\{B_t^2 - t,\ t \geq 0\}$ is a martingale. Therefore, while *both* of these limits are valid ways to define the integral I_t, it is reasonable to use as the definition the limit for which a martingale is produced. And so we make the following definition:

$$\begin{aligned} \int_0^t B_s \, \mathrm{d}B_s &= \lim L_n \text{ in } L^2 \\ &= \frac{1}{2}B_t^2 - \frac{t}{2}. \end{aligned} \tag{11.6}$$