Lecture #13: Calculating Wiener Integrals

Now that we have defined the Wiener integral of a bounded, piecewise continuous deterministic function in $L^2([0,\infty))$ with respect to Brownian motion as a normal random variable, namely

$$\int_0^t g(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right),$$

it might seem like we are done. However, as our ultimate goal is to be able to integrate random functions with respect to Brownian motion, it seems useful to try and develop a *calculus* for Wiener integration. The key computational tool that we will develop is an integration-by-parts formula. But first we need to complete the following exercise.

Exercise 9.1. Verify that the Wiener integral is a linear operator. That is, show that if α, $\beta \in \mathbb{R}$ are constants, and g and h are bounded, piecewise continuous functions in $L^2([0,\infty))$, then

$$\int_0^t [\alpha g(s) + \beta h(s)]\,\mathrm{d}B_s = \alpha \int_0^t g(s)\,\mathrm{d}B_s + \beta \int_0^t h(s)\,\mathrm{d}B_s.$$

Theorem 9.2. *Let $g : [0,\infty) \to \mathbb{R}$ be a bounded, continuous function in $L^2([0,\infty))$. If g is differentiable with g' also bounded and continuous, then the integration-by-parts formula*

$$\int_0^t g(s)\,\mathrm{d}B_s = g(t)B_t - \int_0^t g'(s)B_s\,\mathrm{d}s$$

holds.

Remark. Since all three objects in the above expression are random variables, the equality is interpreted to mean that the distribution of the random variable on the left side and the distribution of the random variable on the right side are the same, namely

$$\mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right).$$

Also note that the second integral on the right side, namely

$$\int_0^t g'(s)B_s\,\mathrm{d}s, \tag{9.1}$$

is the Riemann integral of a function of Brownian motion. Using the notation of the final remark of Lecture #11, we have $h(B_s) = g'(s)B_s$. In Exercise 9.5 you will determine the distribution of (9.1).

Proof. We begin by writing

$$\sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}}) = \sum_{j=1}^n g(t_{j-1})B_{t_j} - \sum_{j=1}^n g(t_{j-1})B_{t_{j-1}}. \tag{9.2}$$

Since g is differentiable, the mean value theorem implies that there exists some value $t_j^* \in [t_{j-1}, t_j]$ such that

$$g'(t_j^*)(t_j - t_{j-1}) = g(t_j) - g(t_{j-1}).$$

Substituting this for $g(t_{j-1})$ in the previous expression (9.2) gives

$$\begin{aligned}\sum_{j=1}^n g(t_{j-1})B_{t_j} - \sum_{j=1}^n g(t_{j-1})B_{t_{j-1}} &= \sum_{j=1}^n g(t_j)B_{t_j} - \sum_{j=1}^n g'(t_j^*)(t_j - t_{j-1})B_{t_j} - \sum_{j=1}^n g(t_{j-1})B_{t_{j-1}} \\ &= \sum_{j=1}^n [g(t_j)B_{t_j} - g(t_{j-1})B_{t_{j-1}}] - \sum_{j=1}^n g'(t_j^*)(t_j - t_{j-1})B_{t_j} \\ &= g(t_n)B_{t_n} - g(t_0)B_{t_0} - \sum_{j=1}^n g'(t_j^*)B_{t_j}(t_j - t_{j-1}) \\ &= g(t)B_t - \sum_{j=1}^n g'(t_j^*)B_{t_j}(t_j - t_{j-1})\end{aligned}$$

since $t_n = t$ and $t_0 = 0$. Notice that we have established an equality between random variables, namely that

$$\sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}}) = g(t)B_t - \sum_{j=1}^n g'(t_j^*)B_{t_j}(t_j - t_{j-1}). \tag{9.3}$$

The proof will be completed if we can show that the distribution of the limiting random variable on the left-side of (9.3) and the distribution of the limiting random variable on the right-side of (9.3) are the same. Of course, we know that

$$\sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}}) \to I_t = \int_0^t g(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right)$$

from our construction of the Wiener integral in last lecture. Thus, we conclude that

$$g(t)B_t - \sum_{j=1}^n g'(t_j^*)B_{t_j}(t_j - t_{j-1}) \to I_t \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right)$$

in distribution as well. We now observe that since g' is bounded and piecewise continuous, and since Brownian motion is continuous, the function $g'(t)B_t$ is necessarily Riemann integrable. Thus,

$$\lim_{n\to\infty} \sum_{j=1}^n g'(t_j^*)B_{t_j}(t_j - t_{j-1}) = \int_0^t g'(s)B_s\,\mathrm{d}s$$

in distribution as in Lecture #11. In other words, we have shown that the distribution of

$$\int_0^t g(s)\,\mathrm{d}B_s$$

and the distribution of

$$g(t)B_t - \int_0^t g'(s)B_s \, \mathrm{d}s$$

are the same, namely

$$\mathcal{N}\left(0, \int_0^t g^2(s) \, \mathrm{d}s\right)$$

and so the proof is complete. □

Example 9.3. Suppose that $t > 0$. It might seem obvious that

$$B_t = \int_0^t \mathrm{d}B_s.$$

However, since Brownian motion is nowhere differentiable, and since we have only defined the Wiener integral as a normal random variable, this equality needs a proof. Since $B_t \sim \mathcal{N}(0, t)$ and since

$$\int_0^t \mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t 1^2 \, \mathrm{d}s\right) = \mathcal{N}(0, t),$$

we conclude that

$$B_t = \int_0^t \mathrm{d}B_s$$

in distribution. Alternatively, let $g \equiv 1$ so that the integration-by-parts formula implies

$$\int_0^t \mathrm{d}B_s = g(t)B_t - \int_0^t g'(s)B_s \, \mathrm{d}s = B_t - 0 = B_t.$$

Example 9.4. Suppose that we choose $t = 1$ and $g(s) = s$. The integration-by-parts formula implies that

$$\int_0^1 s \, \mathrm{d}B_s = B_1 - \int_0^1 B_s \, \mathrm{d}s.$$

If we now write

$$B_1 = \int_0^1 \mathrm{d}B_s$$

and use linearity of the stochastic integral, then we find

$$\int_0^1 B_s \, \mathrm{d}s = B_1 - \int_0^1 s \, \mathrm{d}B_s = \int_0^1 \mathrm{d}B_s - \int_0^1 s \, \mathrm{d}B_s = \int_0^1 (1 - s) \, \mathrm{d}B_s.$$

Since

$$\int_0^1 (1 - s) \, \mathrm{d}B_s$$

is normally distributed with mean 0 and variance

$$\int_0^1 (1 - s)^2 \, \mathrm{d}s = \frac{1}{3},$$

we conclude that

$$\int_0^1 B_s \, \mathrm{d}s \sim \mathcal{N}(0, 1/3).$$

Thus, we have a different derivation of the fact that we proved in Lecture #11.

Exercise 9.5. Show that

$$\int_0^1 g'(s)B_s\,\mathrm{d}s = \int_0^1 [g(1) - g(s)]\,\mathrm{d}B_s$$

where g is any antiderivative of g'. Conclude that

$$\int_0^1 g'(s)B_s\,\mathrm{d}s \sim \mathcal{N}\left(0, \int_0^1 [g(1) - g(s)]^2\,\mathrm{d}s\right).$$

In general, this exercise shows that for fixed $t > 0$, we have

$$\int_0^t g'(s)B_s\,\mathrm{d}s \sim \mathcal{N}\left(0, \int_0^t [g(t) - g(s)]^2\,\mathrm{d}s\right).$$

Exercise 9.6. Use the result of Exercise 9.5 to establish the following generalization of Example 9.4. Show that if $n = 0, 1, 2, \ldots$ is a non-negative integer, then

$$\int_0^1 s^n B_s\,\mathrm{d}s \sim \mathcal{N}\left(0, \frac{2}{(2n+3)(n+2)}\right).$$