Lecture #11: The Riemann Integral of Brownian Motion

Before *integrating with respect to* Brownian motion it seems reasonable to try and *integrate* Brownian motion itself. This will help us get a feel for some of the technicalities involved when the integrand/integrator in a stochastic process.

Suppose that $\{B_t, 0 \le t \le 1\}$ is a Brownian motion. Since Brownian motion is continuous with probability one, it follows from Theorem 6.2 that Brownian motion is Riemann integrable. Thus, at least theoretically, we can integrate Brownian motion, although it is not so clear what the Riemann integral of it is. To be a bit more precise, suppose that $B_t(\omega)$, $0 \le t \le 1$, is a realization of Brownian motion (a so-called *sample path* or *trajectory*) and let

$$I = \int_0^1 B_s(\omega)\,\mathrm{d}s$$

denote the Riemann integral of the function $B(\omega)$ on $[0, 1]$. (By this notation, we mean that $B(\omega)$ is the function and $B(\omega)(t) = B_t(\omega)$ is the value of this function at time t. This is analogous to our notation in calculus in which g is the function and $g(t)$ is the value of this function at time t.)

Question. What can be said about I?

On the one hand, we know from elementary calculus that the Riemann integral represents the area under the curve, and so we at least have that interpretation of I. On the other hand, since Brownian motion is nowhere differentiable with probability one, there is no hope of using the fundamental theorem of calculus to evaluate I. Furthermore, since the value of I depends on the realization $B(\omega)$ observed, we should really be viewing I as a function of ω; that is,

$$I(\omega) = \int_0^1 B_s(\omega)\,\mathrm{d}s.$$

It is now clear that I is itself a random variable, and so the best that we can hope for in terms of "calculating" the Riemann integral I is to determine its distribution.

As noted above, the Riemann integral I necessarily exists by Theorem 6.2, which means that in order to determine its distribution, it is sufficient to determine the distribution of the limit of the right-hand sums

$$I = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} B_{i/n}.$$

(See the final remark of from last lecture.) Therefore, we begin by calculating the distribution of

$$I^{(n)} = \frac{1}{n} \sum_{i=1}^{n} B_{i/n}. \tag{7.1}$$

We know that for each $i = 1, \dots, n$, the distribution of $B_{i/n}$ is $\mathcal{N}(0, i/n)$. The problem, however, is that the sum in (7.1) is not a sum of independent random variables—only Brownian *increments* are independent. However, we can use a little algebraic trick to express this as the sum of independent increments. Notice that

$$\sum_{i=1}^{n} Y_i = nY_1 + (n-1)(Y_2 - Y_1) + (n-2)(Y_3 - Y_2) + \cdots + 2(Y_{n-1} - Y_{n-2}) + (Y_n - Y_{n-1}).$$

We now let $Y_i = B_{i/n}$ so that $Y_i \sim \mathcal{N}(0, i/n)$. Furthermore, $Y_i - Y_{i-1} \sim \mathcal{N}(0, 1/n)$, and the sum above is the sum of independent normal random variables, so it too is normal. Let $X_i = Y_i - Y_{i-1} \sim \mathcal{N}(0, 1/n)$ so that $X_1, X_2, \dots, X_n$ are independent and

$$\sum_{i=1}^{n} Y_i = nX_1 + (n-1)X_2 + \cdots + 2X_{n-1} + X_n = \sum_{i=1}^{n} (n-i+1)X_i \sim \mathcal{N}\left(0, \frac{1}{n}\sum_{i=1}^{n}(n-i+1)^2\right)$$

by Exercise 3.12. Since

$$\sum_{i=1}^{n}(n-i+1)^2 = n^2 + (n-1)^2 + \cdots + 2^2 + 1 = \frac{n(n+1)(2n+1)}{6},$$

we see that

$$\sum_{i=1}^{n} Y_i \sim \mathcal{N}\left(0, \frac{(n+1)(2n+1)}{6}\right),$$

and so finally piecing everything together we have

$$I^{(n)} = \frac{1}{n}\sum_{i=1}^{n} B_{i/n} \sim \mathcal{N}\left(0, \frac{(n+1)(2n+1)}{6n^2}\right) = \mathcal{N}\left(0, \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

Hence, we now conclude that as $n \to \infty$, the variance of $I^{(n)}$ approaches $1/3$ so by Theorem 3.24, the distribution of I is

$$I \sim \mathcal{N}\left(0, \tfrac{1}{3}\right).$$

In summary, this result says that if we consider the area under a Brownian path up to time 1, then that (random) area is normally distributed with mean 0 and variance 1/3. Weird.

Remark. Theorem 6.2 tells us that for any fixed $t > 0$ we can, in theory, "compute" (i.e., determine the distribution of) any Riemann integral of the form

$$\int_0^t h(B_s)\,\mathrm{d}s$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function. Unless h is relatively simple, however, it is not so straightforward to determine the resulting distribution.