

This assignment is due at the beginning of class on Monday, October 27, 2014.

1. Read Chapters 6 and 7, pages 53–72, of Higham.
  - Chapters 6 and 7 develop “geometric Brownian motion” as the model of the stock price movement. Read through these chapters to get a feel for what is to come.
  - Complete the MATLAB exercises from Chapters 6 and 7.
2. (VERY EASY) Suppose that  $\{X_t, t \geq 0\}$  and  $\{Y_t, t \geq 0\}$  are both continuous-time martingales with respect to the filtration  $\{\mathcal{F}_t, t \geq 0\}$ . Show that the process  $\{Z_t, t \geq 0\}$  defined by  $Z_t = X_t + Y_t$  is a continuous-time martingale with respect to the filtration  $\{\mathcal{F}_t, t \geq 0\}$ .
3. The following exercises are from the printed lecture notes.
  - Exercises 5.1, 5.4, 5.5, and 5.6
4. Suppose that  $\{B_t, t \geq 0\}$  is a standard Brownian motion starting at 0, and let  $\{\mathcal{F}_t, t \geq 0\}$  denote the Brownian filtration. Let  $s < t$  and compute  $\mathbb{E}(\sin(B_t) | \mathcal{F}_s)$ . Is  $\{\sin(B_t), t \geq 0\}$  a martingale with respect to  $\{\mathcal{F}_t, t \geq 0\}$ . If not, find a function of  $\sin(B_t)$  that is a martingale.
5. Let  $\{B_t, t \geq 0\}$  be a Brownian motion which, as we know, is a continuous-time martingale with respect to the Brownian filtration  $\{\mathcal{F}_t, t \geq 0\}$ . Let  $t_0 = 0 < t_1 < t_2 < \dots$  be any countable subset of  $[0, \infty)$  and consider  $\{B_{t_j}, j = 0, 1, 2, \dots\}$ . Verify that  $\{B_{t_j}, j = 0, 1, 2, \dots\}$  is a discrete-time martingale with respect to the filtration  $\{\mathcal{F}_{t_j}, j = 0, 1, 2, \dots\}$ . Notationally, you might find it helpful to define  $\{X_j, j = 0, 1, 2, \dots\}$  by setting  $X_j = B_{t_j}$  and show that  $\{X_j, j = 0, 1, 2, \dots\}$  is a discrete-time martingale with respect to the filtration  $\{\mathcal{G}_j, j = 0, 1, 2, \dots\}$  where  $\mathcal{G}_j = \mathcal{F}_{t_j}$ .
6. Suppose that the price of a stock  $\{X_t, t \geq 0\}$  follows a Brownian motion. If the price of the stock at time 3 is \$52, determine the probability that the price of the stock is at least \$55 at time 12.
7. The exchange rate of yen per dollar follows a Brownian motion. The following exchange rates are given.

Day	¥/\$
1	105
2	104
3	102

Calculate the probability that the exchange rate is between 102 and 105 on day 5.

8. If  $Z \sim \mathcal{N}(0, 1)$ , then the process  $\{X_t, t \geq 0\}$  defined by setting  $X_t = \sqrt{t}Z$  is continuous and is marginally distributed as  $\mathcal{N}(0, t)$ . Is  $\{X_t, t \geq 0\}$  a Brownian motion starting at 0?

*(continued)*

**9.** Suppose that  $\{B_t, t \geq 0\}$  and  $\{\tilde{B}_t, t \geq 0\}$  are two independent Brownian motions starting at 0. Let  $\alpha$  be a constant with  $-1 < \alpha < 1$ , and define the process  $\{X_t, t \geq 0\}$  by setting  $X_t = \alpha B_t + \sqrt{1 - \alpha^2} \tilde{B}_t$ . It follows that this process is continuous and is marginally distributed as  $\mathcal{N}(0, t)$ . Is  $\{X_t, t \geq 0\}$  a Brownian motion starting at 0?

**10.** Suppose that  $\{B_t, t \geq 0\}$  is a Brownian motion starting at 0. Let  $\mu$  and  $\sigma > 0$  be real constants. Define the process  $\{X_t, t \geq 0\}$  by setting  $X_t = \sigma B_t + \mu t$ . If  $T > 0$ , show that  $\mathbf{P}\{X_T < 0\} > 0$ . *Hint: What is the (marginal) distribution of  $X_T$ ?*

**11.** The following exercises are from the printed lecture notes.

- Exercises 9.1, 9.5, and 9.6
- Exercises 10.2

**12.** Recall that

$$1_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Let  $g : [0, 1] \rightarrow \mathbb{R}$  be the step function

$$g(t) = 2 \cdot 1_{[0, \frac{1}{4})}(t) - 3 \cdot 1_{[\frac{1}{4}, \frac{5}{8})}(t) + 7 \cdot 1_{[\frac{5}{8}, \frac{3}{4})}(t) + 6 \cdot 1_{[\frac{3}{4}, 1]}(t).$$

(a) Sketch the graph of  $t$  vs.  $g(t)$ .

(b) Compute the Riemann integral  $\int_0^1 g(t) dt$ .

(c) Let  $\{B_t, t \geq 0\}$  be a standard Brownian motion. Determine the distribution of the Wiener integral  $\int_0^1 g(t) dB_t$ .