Statistics 354 (Fall 2018)
Summary of Least Squares Estimation

Suppose that we observe data (x_i, y_i), $i = 1, \dots, n$, and postulate that a simple linear regression model

$$y = \beta_0 + \beta_1 x + \epsilon$$

is appropriate to describe the relationship between x and y. In particular, this assumes that

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \;\; i = 1, \dots, n,$$

where β_0, β_1 are parameters and $\epsilon_1, \dots, \epsilon_n$ are independent with common mean $\mathbb{E}(\epsilon_i) = 0$ and common variance $\mathrm{Var}(\epsilon_i) = \sigma^2$.

Note. We call x_i the explanatory variable and y_i the response variable. We view the explanatory variable as deterministic (i.e., non-random), and we follow the standard statistical practice of viewing y_i as either a random variable or the realization of a random variable (i.e., observed data) depending on the context.

By minimizing

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

with respect to β_0, β_1, we determine that the least squares estimators $\hat{\beta}_0$, $\hat{\beta}_1$ of the parameters β_0, β_1, respectively, are given by

$$\hat{\beta}_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2} = \frac{s_{xy}}{s_{xx}}$$

and

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}.$$

The simple linear regression line is

$$\hat{\mu} = \hat{\beta}_0 + \hat{\beta}_1 x.$$

If $\hat{\mu}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ denotes the ith fitted value, then the ith residual is given by

$$e_i = y_i - \hat{\mu}_i,$$

and we call

$$\sum_{i=1}^{n} e_i^2$$

the residual sum of squares. It can be shown that $\mathbb{E}(\hat{\beta}_0) = \beta_0$ and $\mathbb{E}(\hat{\beta}_1) = \beta_1$ implying that $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators of β_0, β_1, respectively, with variances

$$\mathrm{Var}(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\overline{x}^2}{s_{xx}} \right) \quad \text{and} \quad \mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{s_{xx}}.$$