Stat 354 Fall 2018
Solutions to Assignment #1

1. (a) We showed in class that $\mathbb{E}(\hat\beta_0) = \beta_0$ and $\mathbb{E}(\hat\beta_1) = \beta_1$. This implies that

$$\mathbb{E}(\hat\mu_0) = \mathbb{E}(\hat\beta_0 + \hat\beta_1 x_0) = \mathbb{E}(\hat\beta_0) + x_0\mathbb{E}(\hat\beta_1) = \beta_0 + \beta_1 x_0 = \mu_0$$

as required.

1. (b) We showed in class that $\hat\beta_1$ could be written in the form

$$\hat\beta_1 = \sum \left(\frac{x_i - \overline{x}}{s_{xx}}\right) y_i.$$

Therefore, we can express $\hat\mu_0$ as

$$\begin{aligned}\hat\mu_0 = \hat\beta_0 + \hat\beta_1 x_0 = \overline{y} - \hat\beta_1\overline{x} + \hat\beta_1 x_0 &= \overline{y} + \hat\beta_1(x_0 - \overline{x})\\ &= \frac{1}{n}\left(\sum y_i\right) + (x_0 - \overline{x})\left[\sum\left(\frac{x_i - \overline{x}}{s_{xx}}\right) y_i\right]\\ &= \sum\left[\frac{1}{n} + \frac{(x_0 - \overline{x})(x_i - \overline{x})}{s_{xx}}\right] y_i\end{aligned}$$

Since $y_1, \ldots, y_n$ are independent with $\mathrm{Var}(y_i) = \sigma^2$, we conclude that

$$\begin{aligned}\mathrm{Var}(\hat\mu_0) &= \mathrm{Var}\left(\sum\left[\frac{1}{n} + \frac{(x_0 - \overline{x})(x_i - \overline{x})}{s_{xx}}\right] y_i\right)\\ &= \sum\left[\frac{1}{n} + \frac{(x_0 - \overline{x})(x_i - \overline{x})}{s_{xx}}\right]^2 \mathrm{Var}(y_i)\\ &= \sum\left[\frac{1}{n} + \frac{(x_0 - \overline{x})(x_i - \overline{x})}{s_{xx}}\right]^2 \sigma^2\\ &= \sigma^2\sum\left[\frac{1}{n^2} + 2\frac{(x_0 - \overline{x})(x_i - \overline{x})}{n s_{xx}} + \frac{(x_0 - \overline{x})^2(x_i - \overline{x})^2}{s_{xx}^2}\right]\\ &= \sigma^2\left(\sum\frac{1}{n^2}\right) + 2\frac{(x_0 - \overline{x})\sigma^2}{n s_{xx}}\left[\sum(x_i - \overline{x})\right] + \frac{(x_0 - \overline{x})^2\sigma^2}{s_{xx}^2}\left[\sum(x_i - \overline{x})^2\right]\\ &= \frac{\sigma^2}{n} + 0 + \frac{(x_0 - \overline{x})^2\sigma^2}{s_{xx}^2}\cdot s_{xx}\\ &= \left(\frac{1}{n} + \frac{(x_0 - \overline{x})^2}{s_{xx}}\right)\sigma^2\end{aligned}$$

as required.

1. (c) The easiest way to solve this problem is to substitute in $\hat\beta_0 = \overline{y} - \hat\beta_1\overline{x}$. Doing so yields

$$\begin{aligned}\sum(y_i - \hat\beta_0 - \hat\beta_1 x_i)^2 &= \sum(y_i - \overline{y} + \hat\beta_1\overline{x} - \hat\beta_1 x_i)^2\\ &= \sum\left[(y_i - \overline{y}) - \hat\beta_1(x_i - \overline{x})\right]^2\\ &= \left[\sum(y_i - \overline{y})^2\right] + \hat\beta_1^2\left[\sum(x_i - \overline{x})^2\right] - 2\hat\beta_1\left[\sum(y_i - \overline{y})(x_i - \overline{x})\right]\\ &= s_{yy} - \hat\beta_1^2 s_{xx} + 2\hat\beta_1 s_{xy}.\end{aligned}$$

If we now substitute in $\hat{\beta}_1 = s_{xy}/s_{xx}$, we obtain

$$s_{yy} - \hat{\beta}_1^2 s_{xx} + 2\hat{\beta}_1 s_{xy} = s_{yy} - \left(\frac{s_{xy}}{s_{xx}}\right)^2 s_{xx} + 2\left(\frac{s_{xy}}{s_{xx}}\right) s_{xy} = s_{yy} - \frac{s_{xy}^2}{s_{xx}} = s_{yy} - \hat{\beta}_1^2 s_{xx}$$

as required.

2. (a) Since $y_1, \ldots, y_n$ are independent with $\mathbb{E}(y_i) = \beta_0 + \beta_1 x_i$ and $\mathrm{Var}(y_i) = \sigma^2$, we conclude that $\mathbb{E}(\overline{y}) = \beta_0 + \beta_1 \overline{x}$ and $\mathrm{Var}(\overline{y}) = \sigma^2/n$. Therefore, we obtain

(i) $\mathbb{E}(y_i^2) = \mathrm{Var}(y_i) + [\mathbb{E}(y_i)]^2 = \sigma^2 + (\beta_0 + \beta_1 x_i)^2$, and

(ii) $\mathbb{E}(\overline{y}^2) = \mathrm{Var}(\overline{y}) + [\mathbb{E}(\overline{y})]^2 = \dfrac{\sigma^2}{n} + (\beta_0 + \beta_1 \overline{x})^2$.

Finally,

(iii) $\mathbb{E}(\hat{\beta}_1^2) = \mathrm{Var}(\hat{\beta}_1^2) + [\mathbb{E}(\hat{\beta}_1)]^2 = \dfrac{\sigma^2}{s_{xx}} + \beta_1^2$

using facts that were proved in class (as noted in Problem 1).

2. (b) In order to solve this problem we use the facts (as proved in class) that

$$s_{yy} = \sum (y_i - \overline{y})^2 = \left(\sum y_i^2\right) - n\overline{y}^2 \quad \text{and} \quad s_{xx} = \sum (x_i - \overline{x})^2 = \left(\sum x_i^2\right) - n\overline{x}^2.$$

This implies that

$$\begin{aligned}
\mathbb{E}(s_{yy}) &= \mathbb{E}\left[\left(\sum y_i^2\right) - n\overline{y}^2\right] \\
&= \left[\sum \mathbb{E}(y_i^2)\right] - n\mathbb{E}(\overline{y}^2) \\
&= \left[\sum \left(\sigma^2 + (\beta_0 + \beta_1 x_i)^2\right)\right] - n\left[\frac{\sigma^2}{n} + (\beta_0 + \beta_1 \overline{x})^2\right] \\
&= \left(\sum \sigma^2\right) + \left[\sum (\beta_0 + \beta_1 x_i)^2\right] - \sigma^2 - n(\beta_0 + \beta_1 \overline{x})^2 \\
&= (n-1)\sigma^2 + \left[\sum (\beta_0 + \beta_1 x_i)^2\right] - n(\beta_0 + \beta_1 \overline{x})^2 \\
&= (n-1)\sigma^2 + \left[\sum (\beta_0^2 + 2\beta_0\beta_1 x_i + \beta_1^2 x_i^2)\right] - n(\beta_0^2 + 2\beta_0\beta_1 \overline{x} + \beta_1^2 \overline{x}^2) \\
&= (n-1)\sigma^2 + \left[\left(\sum \beta_0^2\right) - n\beta_0^2\right] + 2\beta_0\beta_1\left[\left(\sum x_i\right) - n\overline{x}\right] + \beta_1^2\left[\left(\sum x_i^2\right) - n\overline{x}^2\right] \\
&= (n-1)\sigma^2 + 0 + 0 + \beta_1^2\left[\left(\sum x_i^2\right) - n\overline{x}^2\right] \\
&= (n-1)\sigma^2 + \beta_1^2\left[\sum (x_i - \overline{x})^2\right] \\
&= (n-1)\sigma^2 + \beta_1^2 s_{xx}
\end{aligned}$$

as required.

2. (c) Using 1.(c) along with 2.(a)(iii) and 2.(b), we now find

$$\begin{aligned}\mathbb{E}\left[\sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2\right] = \mathbb{E}\left(s_{yy} - \hat{\beta}_1^2 s_{xx}\right) &= \mathbb{E}(s_{yy}) - s_{xx}\mathbb{E}(\hat{\beta}_1^2) \\ &= (n-1)\sigma^2 + \beta_1^2 s_{xx} - s_{xx}\left(\frac{\sigma^2}{s_{xx}} + \beta_1^2\right) \\ &= (n-1)\sigma^2 + \beta_1^2 s_{xx} - \sigma^2 - \beta_1^2 s_{xx} \\ &= (n-2)\sigma^2\end{aligned}$$

as required.

2. (d) It now follows from 2.(c) that

$$\mathbb{E}(\hat{\sigma}^2) = \mathbb{E}\left[\frac{1}{n}\sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2\right] = \frac{1}{n}\mathbb{E}\left[\sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2\right] = \frac{1}{n}\cdot(n-2)\sigma^2 = \left(\frac{n-2}{n}\right)\sigma^2$$

as required.

3. As shown in class, the simple linear regression model $y_i = \beta_0 + \beta_1 x + \epsilon$ leads to the normal equations

$$\begin{aligned} n\beta_0 + \beta_1 \sum x_i &= \sum y_i \\ \beta_0 \sum x_i + \beta_2 \sum x_i^2 &= \sum x_i y_i \end{aligned}$$

which have unique solution $\hat{\beta}_0$, $\hat{\beta}_1$. Replacing x_i by kx_i leads to the new simple linear regression model $y_i = \beta_0^{\text{new}} + \beta_1^{\text{new}}(kx) + \epsilon$. The corresponding normal equations are

$$\begin{aligned} n\beta_0^{\text{new}} + \beta_1^{\text{new}} \sum (kx_i) &= \sum y_i \\ \beta_0^{\text{new}} \sum (kx_i) + \beta_1^{\text{new}} \sum (kx_i)^2 &= \sum (kx_i) y_i \end{aligned}$$

which have unique solution $\hat{\beta}_0^{\text{new}}$, $\hat{\beta}_1^{\text{new}}$. If we note that by factoring out appropriate factors of k, the second set of normal equations can be re-written as

$$\begin{aligned} n\beta_0^{\text{new}} + (k\beta_1^{\text{new}}) \sum x_i &= \sum y_i \\ \beta_0^{\text{new}} \sum x_i + (k\beta_1^{\text{new}}) \sum x_i^2 &= \sum x_i y_i \end{aligned}$$

from which we immediately conclude that $\hat{\beta}_0^{\text{new}} = \hat{\beta}_0$ and $\hat{\beta}_1^{\text{new}} = k^{-1}\hat{\beta}_1$ as required.

4. (a) If $S(\beta) = \sum(y_i - \beta x_i)^2$, then

$$S'(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta} S(\beta) = -2\sum x_i(y_i - \beta x_i).$$

The only critical point for $S(\beta)$ occurs when $S'(\beta) = 0$, namely at

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}.$$

Since $S(\beta) > 0$ for all β, we conclude that $\hat{\beta}$ is, in fact, where the minimum of $S(\beta)$ occurs. *Note that, equivalently, one could check that $S''(\hat{\beta}) > 0$.*

4. (b) Since $\mathbb{E}(y_i) = \mathbb{E}(\beta x_i + \epsilon_i) = \beta x_i + \mathbb{E}(\epsilon_i) = \beta x_i$, we deduce that

$$\mathbb{E}(\hat{\beta}) = \mathbb{E}\left(\frac{\sum x_i y_i}{\sum x_i^2}\right) = \frac{\sum x_i \mathbb{E}(y_i)}{\sum x_i^2} = \frac{\sum x_i (\beta x_i)}{\sum x_i^2} = \beta \frac{\sum x_i^2}{\sum x_i^2} = \beta$$

so that $\hat{\beta}$ is, in fact, an unbiased estimator of β.

4. (c) The fact that $\epsilon_1, \ldots, \epsilon_n$ are independent implies that $y_1, \ldots, y_n$ are independent. Therefore, since $\mathrm{Var}(y_i) = \mathrm{Var}(\beta x_i + \epsilon_i) = \mathrm{Var}(\epsilon_i) = \sigma^2$, we deduce that

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}\left(\frac{\sum x_i y_i}{\sum x_i^2}\right) = \frac{\sum x_i^2 \,\mathrm{Var}(y_i)}{(\sum x_i^2)^2} = \frac{\sum x_i^2 \sigma^2}{(\sum x_i^2)^2} = \sigma^2 \frac{\sum x_i^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}$$

as required.