Stat 352: Solutions to Assignment #5

1. (a) An expression for the posterior density is

$$f(\theta|y) = \frac{f(y|\theta)g(\theta)}{\int_{-\infty}^{\infty} f(y|\theta)g(\theta)\,\mathrm{d}\theta} = \frac{\frac{1}{\theta}\exp\{-\theta^2 - y^2/\theta\}}{\int_0^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - y^2/\theta\}\,\mathrm{d}\theta}.$$

1. (b) If $y = 1$, then

$$f(\theta|y=1) = \frac{\frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}}{\int_0^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}\,\mathrm{d}\theta}.$$

Using MAPLE

```
> evalf(Int(exp(-x^2-1/x)/x, x=0..infinity));
>                 0.1869287323
```

we find

$$\int_0^{\infty} \frac{1}{\theta}\exp\{-\theta^2 - 1/\theta\}\,\mathrm{d}\theta = 0.1869287323$$

and so

$$f(\theta|y=1) = \frac{1}{0.1869287323\theta} \cdot \exp\{-\theta^2 - 1/\theta\} = \frac{5.349632385}{\theta} \cdot \exp\{-\theta^2 - 1/\theta\}, \quad \theta > 0.$$

1. (c) We begin by noticing that the prior density for θ is the absolute value of a $\mathcal{N}(0, 1/2)$ random variable. Thus, we can simulate from the prior density, and so we can use the prior as our envelope. In order to implement the envelope method we change variables. That is, our goal is to sample the random variable X having density

$$f(x) = \frac{5.349632385}{x} \cdot \exp\{-x^2 - 1/x\}, \quad x > 0.$$

We use the enveloping function

$$g(y) = \frac{2}{\sqrt{\pi}} e^{-y^2}, \quad y > 0,$$

and note that $f(x) \le a g(x)$ with $a = 1.8$.

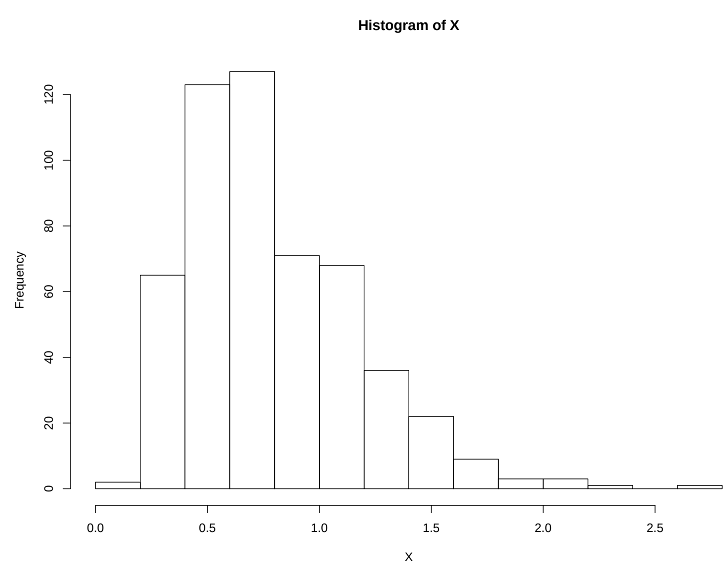
```
> y <- abs(rnorm(1,0,1/sqrt(2))) ; runif(1,0, 1.8*2/sqrt(pi)*exp(-y^2) )
[1] 0.3312523

> 5.349632385/y * exp(-y^2 -1/y)
[1] 0.4203123

> y
[1] 1.234151
```

Thus, our simulated value of X is $X = 1.234151$.

1. (d) Running the following code gave a random sample of size 531 (out of a possible 1000) from the density $f(x)$.

```
> Y=0
> N=1000
> y <- abs(rnorm(N,0,1/sqrt(2)))
> u <- runif(N,0,1.8*2/sqrt(pi)*exp(-y^2))
> f <- 5.349632385/y * exp(-y^2 -1/y)
> m=pmax(f-u,0)
> for(i in 1:N) ifelse(m[i]==0, Y[i]<-NA,Y[i]<-y[i])
> tmp <-na.omit(Y)
> X=0
> for (i in 1:length(tmp)) X[i]<-tmp[i]
> hist(X)
```

The plot of the histogram is as follows.



2. (a) Let the order of the rows and columns be strawberry, chocolate, vanilla.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}.$$

Since all of the entries of $\mathbf{P} = \mathbf{P}^1$ are non-zero, this Markov chain *is* regular.

2. (b) The equilibrium vector is found by solving $\bar{\pi}\mathbf{P} = \bar{\pi}$ where $\bar{\pi} = [\pi_1\ \pi_2\ \pi_3]$, and $\pi_1+\pi_2+\pi_3 = 1$. The corresponding system is therefore

$$\begin{aligned} \pi_1 + \quad \pi_2 + \quad \pi_3 &= 1, \\ 0.5\pi_1 + 0.2\pi_2 + 0.1\pi_3 &= \pi_1, \\ 0.1\pi_1 + 0.3\pi_2 + 0.2\pi_3 &= \pi_2, \\ 0.4\pi_1 + 0.5\pi_2 + 0.7\pi_3 &= \pi_3. \end{aligned}$$

Solving this system of equations gives

$$\bar{\pi} = \left[\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right].$$

2. (c) In order to determine the probability that Jay's grandchild will prefer vanilla ice-cream assuming Jay prefers chocolate ice-cream, we simply compute $\mathbf{P}^2$ which is

$$\mathbf{P}^2 = \begin{bmatrix} 0.31 & 0.16 & 0.53 \\ 0.21 & 0.21 & 0.58 \\ 0.16 & 0.21 & 0.63 \end{bmatrix}.$$

The required probability is the entry $(2,3)$ of $\mathbf{P}^2$ which is 0.58.

3. The transition matrix for this Markov chain is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}$$

and so we can determine the long run behaviour of the chain by taking high powers of the matrix. Using MAPLE, we see that

$$\mathbf{P}^n \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix}$$

Since we are interested in the probability of absorption in state A assuming the chain starts in state C, the required probability is entry $(3,1)$ of the limiting matrix which is 0.2.

4. Using Maple to compute $\mathbf{P}^{23}$ gives

$$\mathbf{P}^{23} = \begin{bmatrix} 0.3787878788 & 0.2575757576 & 0.3636363636 \\ 0.3787878788 & 0.2575757576 & 0.3636363636 \\ 0.3787878788 & 0.2575757576 & 0.3636363636 \end{bmatrix}.$$

Thus, the long run probability that the Markov chain is in state 1 is approximately 0.2576.

Solving $\bar{\pi}\mathbf{P} = \bar{\pi}$ where $\bar{\pi} = [\pi_0, \pi_1, \pi_2]$ gives the following system of equations:

$$\begin{aligned} 0.4\pi_0 + 0.6\pi_1 + 0.2\pi_2 &= \pi_0, \\ 0.2\pi_0 \qquad\qquad + 0.5\pi_2 &= \pi_1, \\ 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2 &= \pi_2. \end{aligned}$$

Row reducing yields

$$24\pi_0 - 25\pi_2 = 0 \quad \text{and} \quad 24\pi_1 - 17\pi_2 = 0$$

and so using the fact that $\pi_0 + \pi_1 + \pi_2 = 1$ we conclude

$$\frac{25}{24}\pi_2 + \frac{17}{24}\pi_2 + \pi_3 = 2.$$

Thus,

$$\bar{\pi} = [\pi_0 \ , \ \pi_1 \ , \ \pi_2] = \left[\frac{25}{66} \ , \ \frac{17}{66} \ , \ \frac{24}{66}\right]$$

and so we conclude that the long run probability that the Markov chain is in state 1 is $\frac{17}{66} \approx 0.2576$.