Stat 352: Solutions to Assignment #3

1. By Bayes' theorem, we have

$$P(\text{you have AIDS}|\text{test positive}) = \frac{P(\text{test positive}|\text{you have AIDS}) \cdot P(\text{you have AIDS})}{P(\text{test positive})}.$$

We now use the information given in the problem, but need to be careful about the wording. We are told that $P(\text{you have AIDS}) = 1/10000 = 0.0001$ and $P(\text{test positive}|\text{you have AIDS}) = 0.99$. However, the 5% false positive means $P(\text{test positive}|\text{you do NOT have AIDS}) = 0.05$. Therefore, we must calculate $P(\text{test positive})$ using the law of total probability. Thus,

$$\begin{aligned} &P(\text{test positive}) \\ &\quad = P(\text{test positive}|\text{you have AIDS}) \cdot P(\text{you have AIDS}) \\ &\qquad + P(\text{test positive}|\text{you do NOT have AIDS}) \cdot P(\text{you do NOT have AIDS}) \\ &\quad = 0.99 \cdot 0.0001 + 0.05 \cdot 0.9999 \\ &\quad = 0.050094 \end{aligned}$$

so that

$$P(\text{you have AIDS}|\text{test positive}) = \frac{0.99 \times 0.0001}{0.050094} = \frac{1}{506} \approx 0.001976.$$

Notice that this answer is significantly lower than 99%. Are you surprised?

2. (a) In order to maximize the likelihood function, we attempt to maximize the log-likelihood function

$$\ell(\theta) = \left(\sum y_i\right) \log\theta - n\theta - \log y!.$$

We find that

$$\ell'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\ell(\theta) = \frac{\sum y_i}{\theta} - n$$

so that setting $\ell'(\theta) = 0$ implies that $\theta = \overline{y}$. Since

$$\ell''(\theta) = -\frac{\sum y_i}{\theta^2} < 0$$

for all θ, the second derivative test implies

$$\hat{\theta}_{\text{MLE}} = \overline{Y}.$$

2. (b) If we let

$$u = \overline{y}, \quad q(y_1, \dots y_n) = \frac{1}{\prod y_i!}, \quad \text{and} \quad p(u,\theta) = e^{n\theta}\theta^{nu},$$

then since

$$f(y_1, \dots, y_n|\theta) = q(y_1, \dots, y_n) \cdot p(u, \theta)$$

we conclude from the factorization theorem that $U = \overline{Y} = \hat{\theta}_{\text{MLE}}$ is a sufficient statistic for the estimation of θ.

2. (c) Since

$$\log f(y|\theta) = y\log(\theta) - y - \log(y!)$$

we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{y}{\theta} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{y}{\theta^2}.$$

Thus,

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{E(Y)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}.$$

2. (d) Since $I(\theta) = \theta^{-1}$ we conclude that the Jeffreys prior for θ satisfies

$$g(\theta) \propto \frac{1}{\sqrt{\theta}}.$$

2. (e) If the observed data values are $\{1, 0, 2, 4, 3, 0\}$, then the likelihood function is

$$f(y_1 = 1, y_2 = 0, y_3 = 2, y_4 = 4, y_5 = 3, y_6 = 0|\theta) = \frac{e^{-6\theta}\theta^1 \cdot \theta^0 \cdot \theta^2 \cdot \theta^4 \cdot \theta^3 \cdot \theta^0}{1! \cdot 0! \cdot 2! \cdot 4! \cdot 3! \cdot 0!} = \frac{e^{-6\theta}\theta^{10}}{288}.$$

Therefore, the resulting posterior satisfies

$$f(\theta|y_1 = 1, y_2 = 0, y_3 = 2, y_4 = 4, y_5 = 3, y_6 = 0) \propto \frac{1}{\sqrt{\theta}} \cdot e^{-6\theta}\theta^{10} = \theta^{9.5}e^{-6\theta}.$$

Since

$$\int_0^\infty \theta^{9.5}e^{-6\theta}\,\mathrm{d}\theta = \frac{\Gamma(10.5)}{6^{10.5}}$$

we conclude that

$$f(\theta|y_1 = 1, y_2 = 0, y_3 = 2, y_4 = 4, y_5 = 3, y_6 = 0) = \frac{6^{10.5}}{\Gamma(10.5)}\theta^{9.5}e^{-6\theta}, \quad \theta > 0.$$

The posterior mean is therefore

$$\mathbb{E}(\theta|y_1 = 1, y_2 = 0, y_3 = 2, y_4 = 4, y_5 = 3, y_6 = 0) = \frac{10.5}{6}.$$

2. (f) The 90% equal-tailed credible interval is given by $[L, R]$ where L and R satisfy

$$\int_0^L \frac{6^{10.5}}{\Gamma(10.5)}\theta^{9.5}e^{-6\theta}\,\mathrm{d}\theta = 0.05$$

and

$$\int_0^R \frac{6^{10.5}}{\Gamma(10.5)}\theta^{9.5}e^{-6\theta}\,\mathrm{d}\theta = 0.95.$$

Using `R` with the commands `qgamma(0.05, 10.5, 6, 1/6)` and `qgamma(0.95, 10.5, 6, 1/6)` gives

$$[0.9659421\,,\,2.722548]$$

as the required credible interval.

3. Suppose that Y is a random variable with density $f(y|\theta)$ where θ is an unknown parameter. Suppose further that $q(\theta)$ is the prior density for θ corresponding to posterior density $f_q(\theta|y)$ so that

$$f_q(\theta|y) = \frac{f(y|\theta)q(\theta)}{\int f(y|\theta)q(\theta)\,\mathrm{d}\theta}.$$

If $p(\theta)$ is chosen so that $g(\theta) = p(\theta)q(\theta)$ is a legitimate density, then the posterior corresponding to $g(\theta)$ is

$$f(\theta|y) = \frac{f(y|\theta)g(\theta)}{\int f(y|\theta)g(\theta)\,\mathrm{d}\theta} = \frac{f(y|\theta)p(\theta)q(\theta)}{\int f(y|\theta)g(\theta)\,\mathrm{d}\theta}.$$

However,

$$q(\theta) = \frac{f_q(\theta|y)\int f(y|\theta)q(\theta)\,\mathrm{d}\theta}{f(y|\theta)}$$

so that

$$f(\theta|y) = \frac{\int f(y|\theta)q(\theta)\,\mathrm{d}\theta}{\int f(y|\theta)g(\theta)\,\mathrm{d}\theta}\cdot p(\theta)f_q(\theta|y)$$

as required. Note that

$$\frac{\int f(y|\theta)q(\theta)\,\mathrm{d}\theta}{\int f(y|\theta)g(\theta)\,\mathrm{d}\theta}$$

is a constant in θ so that

$$f(\theta|y) \propto p(\theta)f_q(\theta|y).$$

4. We begin by writing

$$f(y|\theta) = (\theta+1)\exp\{\theta\log y\},\quad 0 < y < 1,$$

which shows that $f(y|\theta)$ belongs to an exponential family.

Our guess for the conjugate prior is

$$g(\theta) \propto (\theta+1)^{\delta}e^{\gamma\theta},\quad \theta > -1,$$

where δ, γ are suitably chosen constants.

Note that in order for $g(\theta)$ to be a legitimate density, we must determine the value of

$$\int_{-1}^{\infty}(\theta+1)^{\delta}e^{\gamma\theta}\,\mathrm{d}\theta.$$

Make the change-of-variables $u = \theta + 1$ so that

$$\int_{-1}^{\infty}(\theta+1)^{\delta}e^{\gamma\theta}\,\mathrm{d}\theta = \int_{0}^{\infty}u^{\delta}e^{\gamma(u-1)}\,\mathrm{d}u = e^{-\gamma}\int_{0}^{\infty}u^{\delta}e^{\gamma u}\,\mathrm{d}u,$$

and notice that

$$\int_{0}^{\infty}u^{\delta}e^{\gamma u}\,\mathrm{d}u$$

looks like a gamma function. This integral will converge provided that $\delta > -1$ and $\gamma < 0$. Therefore, let $\alpha = \delta + 1$ and $\beta = -\gamma$ with $\alpha > 0$ and $\beta > 0$ so that

$$\int_{0}^{\infty}u^{\delta}e^{\gamma u}\,\mathrm{d}u = \int_{0}^{\infty}u^{\alpha-1}e^{-\beta u}\,\mathrm{d}u = \frac{\Gamma(\alpha)}{\beta^{\alpha}} = \frac{\Gamma(\delta+1)}{(-\gamma)^{\delta+1}}.$$

Thus,

$$g(\theta) = e^{\gamma} \frac{(-\gamma)^{\delta+1}}{\Gamma(\delta+1)} (\theta+1)^{\delta} e^{\gamma\theta} = \frac{(-\gamma)^{\delta+1}}{\Gamma(\delta+1)} (\theta+1)^{\delta} e^{\gamma(\theta+1)}, \quad \theta > -1,$$

provided that $\delta > -1$ and $\gamma < 0$.

The corresponding posterior satisfies

$$f(\theta|y) \propto f(y|\theta) g(\theta) \propto (\theta+1)^{\delta+1} e^{(\gamma+\log y)\theta}, \quad \theta > -1,$$

provided that $\delta > -1$ and $\gamma < 0$. Note that the posterior $f(\theta|y)$ has the same functional form as $g(\theta)$ which verifies that $g(\theta)$ is, in fact, a conjugate prior.

Furthermore, by mimicking the calculation done above, it is possible write the exact expression for $f(\theta|y)$, namely

$$f(\theta|y) = e^{\gamma+\log y} \frac{(-\gamma-\log y)^{\delta+2}}{\Gamma(\delta+2)} (\theta+1)^{\delta+1} e^{(\gamma+\log y)\theta} = \frac{(-\gamma-\log y)^{\delta+2}}{\Gamma(\delta+2)} (\theta+1)^{\delta+1} e^{(\gamma+\log y)(\theta+1)},$$

for $\theta > -1$ provided that $\delta > -1$ and $\gamma < 0$ (and noting that $\log y < 0$ since $0 < y < 1$).

5. This problem can be solved using the `R` macro `binodp` provided with the text by Bolstad. Entering the commands

```
> theta=c(0.10, 0.24, 0.33, 0.59, 0.68, 0.87)
> prior=c(0.27, 0.17, 0.12, 0.38, 0.05, 0.01)
> binodp(13,25,pi=theta,pi.prior=prior,ret=TRUE)
```

returns the following table

```
     Prior   Likelihood    Posterior
0.1    0.27 3.965539e-08 7.641275e-07
0.24   0.17 2.877309e-04 5.544343e-03
0.33   0.12 2.810532e-03 5.415668e-02
0.59   0.38 4.680523e-02 9.018991e-01
0.68   0.05 1.992572e-03 3.839525e-02
0.87   0.01 1.981978e-07 3.819113e-06
```

where, as noted in lab, the column labelled `Likelihood` should read `Joint`.

Thus, we can summarize the required posterior probabilities as

$$\begin{aligned}
f(\theta = 0.10 \,|\, y = 13) &= 0.0000007641275, \\
f(\theta = 0.24 \,|\, y = 13) &= 0.005544343, \\
f(\theta = 0.33 \,|\, y = 13) &= 0.05415668, \\
f(\theta = 0.59 \,|\, y = 13) &= 0.9018991, \\
f(\theta = 0.68 \,|\, y = 13) &= 0.03839525, \\
f(\theta = 0.01 \,|\, y = 13) &= 0.000003819113.
\end{aligned}$$

Notice that the vast majority of posterior weight is given to $\theta = 0.59$. This is not a surprise since there were $y = 13$ successes in $n = 25$ trials—we would expect the success probability to be about 1/2.