Stat 352 Winter 2008
Assignment #3

This assignment is due at the beginning of class on Thursday, February 7, 2008.

1. Donated blood is screened for AIDS. Suppose that the test has 99% accuracy, and that one in ten thousand people in your age group are HIV positive. The test has a 5% false positive rating as well. Suppose the test screens you as positive. What is the probability that you have AIDS? Is it 99%? (*Hint:* 99% accuracy refers to $P\{\text{test is positive}|\text{ you have AIDS}\}$. You want to determine $P\{\text{you have AIDS}|\text{test is positive}\}$

2. Recall that a random variable Y is said to be Poisson with parameter θ if it has density function

$$f(y|\theta) = \frac{\theta^y e^{-\theta}}{y!}, \quad y = 0, 1, 2, \ldots.$$

Suppose that the random variables $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Poisson(θ) where the parameter θ is unknown. As shown on Assignment #1, the likelihood function is

$$f(y_1, y_2, \ldots, y_n|\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \frac{1}{\prod y_i!} e^{-n\theta} \theta^{\sum y_i}$$

where the sum and product both run from $i = 1$ to n.

(a) STAT 252: Compute $\hat{\theta}_{\text{MLE}}$, the maximum likelihood estimator of θ based on $Y_1, \ldots, Y_n$.

(b) STAT 252: Show that $\hat{\theta}_{\text{MLE}}$ is sufficient for the estimation of θ.

(c) STAT 252: Determine $I(\theta)$, the Fisher information in a single observation from the Poisson(θ) density.

(d) Determine the Jeffreys prior for θ.

(e) Suppose that a sample of size $n = 6$ produces data values $\{1, 0, 2, 4, 3, 0\}$. Determine the posterior distribution for θ using the Jeffreys prior and these data values. What is the posterior mean?

(f) Determine the Bayesian equal-tail 90% credible interval for θ using the Jeffreys prior.

3. Suppose that Y is a random variable with density $f(y|\theta)$ where θ is an unknown parameter. Suppose further that $q(\theta)$ is a prior density for θ which corresponds to posterior density $f_q(\theta|y)$. Let $p(\theta)$ be a function of θ such that $g(\theta) = p(\theta)q(\theta)$ is a legitimate density for θ. Show that the posterior density for θ based on $g(\theta)$ is

$$f(\theta|y) = \frac{\int f(y|\theta)q(\theta)\,\mathrm{d}\theta}{\int f(y|\theta)g(\theta)\,\mathrm{d}\theta} \cdot p(\theta) f_q(\theta|y).$$

(In particular, this shows that $f(\theta|y) \propto p(\theta)f_q(\theta|y)$ and illustrates the logical difficulty with our definition of conjugate family.)

4. Suppose that the random variable Y has density function $f(y|\theta) = (\theta+1)y^\theta$ for $0 < y < 1$ where $\theta > -1$ is a parameter. As shown in class, $f(y|\theta)$ belongs to an exponential family. Determine a conjugate prior density for θ and prove your answer is correct by determining the posterior distribution. (In fact, there are infinitely many conjugate priors. You need to pick one. If you are feeling ambitious you can find the general formula for all priors—it really is not much harder.)

5. Suppose that the distribution of Y is binomial with parameters $n = 25$ and θ where $0 < \theta < 1$ is an unknown parameter. If the prior density for θ is given by

$$g(\theta) = \begin{cases} 0.27, & \text{if } \theta = 0.10, \\ 0.17, & \text{if } \theta = 0.24, \\ 0.12, & \text{if } \theta = 0.33, \\ 0.38, & \text{if } \theta = 0.59, \\ 0.05, & \text{if } \theta = 0.68, \\ 0.01, & \text{if } \theta = 0.87, \end{cases}$$

determine the posterior probabilities if $y = 13$ is observed.