Statistics 351 Fall 2015 Midterm #1 – Solutions

1. (a) We find

$$\frac{1}{c} = \int_1^2 \int_1^y x \,\mathrm{d}x \,\mathrm{d}y = \int_1^2 \left[\frac{x^2}{2}\right]_{x=1}^{x=y} \mathrm{d}y = \int_1^2 \left(\frac{y^2}{2} - \frac{1}{2}\right) \mathrm{d}y = \left[\frac{y^3}{6} - \frac{y}{2}\right]_{y=1}^{y=2}$$
$$= \left(\frac{8}{6} - \frac{2}{2} - \frac{1}{6} + \frac{1}{2}\right)$$
$$= \frac{2}{3}$$

so that $c = 3/2$.

1. (b) By definition,

$$f_X(x) = \int_x^2 cx \,\mathrm{d}y = cx(2-x) = \frac{3}{2}x(2-x)$$

provided that $1 \le x \le 2$.

1. (c) By definition,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{cx}{cx(2-x)} = \frac{1}{2-x}$$

provided that $x \le y \le 2$. Note that $Y|X = x$ is uniformly distribution on $[x, 2]$.

1. (d) Since $Y|X = x \in U[x, 2]$, we know $\mathbb{E}(Y|X = x) = (2 + x)/2$. Equivalently, we find

$$\mathbb{E}(Y|X = x) = \int_x^2 y \cdot \frac{1}{2-x} \,\mathrm{d}y = \frac{1}{2-x}\left[\frac{y^2}{2}\right]_{y=x}^{y=2} = \frac{1}{2-x}\left[2 - \frac{x^2}{2}\right] = \frac{2+x}{2}.$$

2. If $U = X + 2Y$ and $V = X$, then solving for X and Y gives $X = V$ and $Y = (U - V)/2$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$

We now need to be careful with the limits of integration. Since $x > 0$ we see that necessarily $v > 0$. However, $y > 0$ implies that $(u - v)/2 > 0$ so $u - v > 0$ or, equivalently, $u > v$. Therefore, we conclude that for $0 < v < u < \infty$, we have

$$f_{U,V}(u, v) = f_{X,Y}(v, (u - v)/2) \cdot |J| = \frac{v^3}{3}e^{-u} \cdot \frac{1}{2} = \frac{v^3}{6}e^{-u}.$$

The marginal for U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v) \,\mathrm{d}v = \int_0^u \frac{v^3}{6}e^{-u} \,\mathrm{d}v = \frac{e^{-u}}{6}\int_0^u v^3 \,\mathrm{d}v = \frac{e^{-u}}{6}\left[\frac{v^4}{4}\right]_{v=0}^{v=u} = \frac{u^4}{24}e^{-u}$$

provided $u > 0$. Note that $U \in \Gamma(5, 1)$.

3. Using the law of total probability,

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_{-\infty}^{\infty} f_{Y|X=x}(y) f_X(x)\,\mathrm{d}x &= \int_y^1 6x(1-x)\cdot\frac{1}{x}\,\mathrm{d}x \\ &= \int_y^1 6(1-x)\,\mathrm{d}x \\ &= -3(1-x)^2\Big|_{x=y}^{x=1} \\ &= 3(1-y)^2 \end{aligned}$$

provided $0 \le y \le 1$.

4. If $Y = F_X(X)$, then the distribution function of Y is

$$\begin{aligned} P(Y \le y) = P(F_X(X) \le y) = P\left(\frac{1}{2} + \frac{1}{\pi}\arctan(X) \le y\right) &= P(X \le \tan(\pi(y-1/2))) \\ &= \int_{-\infty}^{\tan(\pi(y-1/2))} \frac{1}{\pi}\frac{1}{1+t^2}\,\mathrm{d}t. \end{aligned}$$

The density function of Y is

$$\begin{aligned} f_Y(y) = F_Y'(y) &= \frac{1}{\pi}\frac{1}{1+\tan^2(\pi(y-1/2))}\cdot\frac{\mathrm{d}}{\mathrm{d}y}\tan(\pi(y-1/2)) \\ &= \frac{1}{\pi}\frac{1}{\sec^2(\pi(y-1/2))}\cdot\sec^2(\pi(y-1/2))\cdot\pi \\ &= 1 \end{aligned}$$

provided that $0 \le y \le 1$. Thus, Y is uniformly distributed on $[0,1]$.