Statistics 351 Midterm #2 – November 20, 2015

This exam has 4 problems and is worth a total of 50 points.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For problems with multiple parts, all parts are equally weighted.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

You must answer all of the questions in the exam booklet provided.

1. (*12 points*) Suppose that X_1, X_2, X_3 are independent uniform random variables on the interval $[0, 1]$ so that their common density function is

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

As always, let $X_{(j)}$, $j = 1, 2, 3$, denote the jth order variable.

(a) Determine the joint density of $X_{(1)}, X_{(2)}$.

(b) Compute $P(X_{(2)} < 2X_{(1)})$.

2. (*14 points*) Suppose that X_1 and X_2 are independent exponential random variables with parameter 1 so that their common density function is

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

As usual, let $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$. If $U = X_{(2)} - X_{(1)}$ and $V = X_{(1)}$, determine the joint density of $(U, V)'$ and deduce that U and V are independent.

3. (*12 points*) Suppose that X_1 and X_2 are independent $N(0, 1)$ random variables. Define the random variables Y_1, Y_2, and Y_3 by setting

$$Y_1 = X_1 - 2X_2 \quad \text{and} \quad Y_2 = 2X_1 + 3X_2 - 1 \quad \text{and} \quad Y_3 = X_2 + 2.$$

Determine the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$.

4. (*12 points*) As you know from Stat 251, if the random variables X and Y are independent, then they are necessarily uncorrelated. You were also told that, in general, the converse is false; namely, there are examples of dependent random variables X and Y which are uncorrelated. The purpose of this problem is to prove that in the case of the multivariate normal distribution, the components of the random vector $\mathbf{X}$ are independent if and only if they are uncorrected.

Suppose that $\mathbf{X} = (X, Y)'$ has a multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

where $\rho = \operatorname{corr}(X, Y) \in (-1, 1)$.

- **(a)** Write down the density function for $\mathbf{X}$.
- **(b)** Prove that if $\rho = 0$, then X and Y are independent.