Statistics 351 Midterm #1 – October 21, 2015

This exam has 4 problems and is worth a total of 50 points.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For problems with multiple parts, all parts are equally weighted.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

You must answer all of the questions in the exam booklet provided.

1. (*16 points*) Suppose that the random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} cx, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

where the value of the normalizing constant c is chosen so that $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$.

(a) Determine the value of c.

(b) Determine $f_X(x)$, the marginal density function of X.

(c) Calculate $f_{Y|X=x}(y)$, the conditional density function of Y given $X = x$.

(d) Calculate $\mathbb{E}(Y|X = x)$, the conditional expectation of Y given $X = x$.

2. (*14 points*) Suppose that the two-dimensional random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \frac{x^3}{3}e^{-x-2y}$$

provided that $x > 0$ and $y > 0$. If the random vector $(U, V)'$ is defined by setting

$$U = X + 2Y \quad \text{and} \quad V = X,$$

determine $f_U(u)$, the marginal density function of U.

(continued)

3. (*10 points*) Suppose that $(X, Y)'$ is a two-dimensional random vector. It is known that the marginal distribution of X is beta with parameters 2 and 2 so that the density of X is

$$f_X(x) = 6x(1-x)$$

for $0 \le x \le 1$. It is also known that the conditional distribution of Y given $X = x$ is uniform on $[0, x]$ so that

$$f_{Y|X=x}(y) = \frac{1}{x}$$

for $0 \le y \le x$. Determine $f_Y(y)$, the density function for Y.

4. (*10 points*) Suppose that X has a Cauchy distribution so that the density function of X is

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

for $-\infty < x < \infty$ and distribution function of X is

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$$

for $-\infty < x < \infty$. Consider the random variable

$$Y = F_X(X) = \frac{1}{2} + \frac{1}{\pi}\arctan(X).$$

(In other words, Y is defined by evaluating the distribution function of X at X itself.) Determine the density function of Y.

Hint: Recall that $\frac{\mathrm{d}}{\mathrm{d}\theta}\tan(\theta) = \sec^2(\theta)$. The trig identity $1 + \tan^2(\theta) = \sec^2(\theta)$ may be useful.