Statistics 351 Fall 2008 Midterm #2 – Solutions

1. (a) Clearly $\boldsymbol{\mu} = (0,0)'$. As for $\boldsymbol{\Lambda}$, observe that

$$-\frac{1}{2}x_1^2 + x_1x_2 - x_2^2 = -\frac{1}{2}(x_1^2 - 2x_1x_2 + 2x_2^2)$$

so that

$$\boldsymbol{\Lambda}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{which implies that} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

since $\det[\boldsymbol{\Lambda}] = \det[\boldsymbol{\Lambda}^{-1}] = 1$.

1. (b) Let

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. Since

$$B\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B\boldsymbol{\Lambda}B' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

we conclude from Theorem V.3.1 that

$$\mathbf{Y} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}\right).$$

1. (c) Since $\mathbf{Y} = (Y_1, Y_2)'$ is multivariate normal, we know from equation (6.1) on page 129 that $Y_2|Y_1 = y_1$ is normal with

$$E(Y_2|Y_1 = y_1) = \mu_{y_2} + \rho\frac{\sigma_{y_2}}{\sigma_{y_1}}(y_1 - \mu_{y_1}) \quad \text{and} \quad \text{var}(Y_2|Y_1 = y_1) = \sigma_{y_2}^2(1-\rho^2).$$

Since $Y_1 \in N(0,1)$, $Y_2 \in N(0,5)$ and $\text{cov}(Y_1, Y_2) = 1$ so that $\rho = \text{corr}(Y_1, Y_2) = \frac{1}{\sqrt{5}}$, we conclude that $Y_2|Y_1 = 0 \in N(0,4)$.

2. (a) Since

$$\begin{aligned}\det[\boldsymbol{\Lambda} - \lambda I] = \left(\frac{7}{4} - \lambda\right)\left(\frac{5}{4} - \lambda\right) - \frac{3}{16} &= \frac{35}{16} - \frac{5}{4}\lambda - \frac{7}{4}\lambda + \lambda^2 - \frac{3}{16} \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2)\end{aligned}$$

we conclude that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

2. (b) Since

$$\boldsymbol{\Lambda} - \lambda_1 I = \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \sim \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -\sqrt{3} \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{bmatrix}$$

we conclude that an eigenvector corresponding to λ_1 is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}.$$

Since

$$\mathbf{\Lambda} - \lambda_2 I = \begin{bmatrix} -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

we conclude that an eigenvector corresponding to λ_2 is

$$\mathbf{v}_2 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}.$$

Thus, the required orthogonal matrix C and required diagonal matrix D are

$$C = \begin{bmatrix} \frac{\mathbf{v}_1}{||\mathbf{v}_1||} & \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

2. (c) Since $\det[\mathbf{\Lambda}] = 35/16 - 3/16 = 32/16 = 2$ and

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} \frac{5}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & \frac{7}{8} \end{bmatrix}$$

so that

$$\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = \frac{5}{8}x_1^2 - \frac{\sqrt{3}}{4}x_1 x_2 + \frac{7}{8}x_2^2$$

we conclude that the density function of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{\det[\mathbf{\Lambda}]}} \exp\left\{-\frac{1}{2}\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}\right\} = \frac{1}{2\pi\sqrt{2}} \exp\left\{-\frac{1}{2}\left(\frac{5}{8}x_1^2 - \frac{\sqrt{3}}{4}x_1 x_2 + \frac{7}{8}x_2^2\right)\right\}.$$

2. (d) If $\mathbf{Y} = C'\mathbf{X}$, then by Theorem V.8.1 (or Theorem V.3.1) $\mathbf{Y}$ has a multivariate normal distribution with mean vector

$$C'\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and covariance matrix

$$C'\mathbf{\Lambda}C = C'(CDC')C = (C'C)D(C'C) = IDI = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

2. (e) Since Y_1 and Y_2 are components of a multivariate normal random vector and satisfy $\operatorname{cov}(Y_1, Y_2) = 0$, we know from Theorem V.7.1 that Y_1 and Y_2 are independent.

2. (f) It follows from Theorem V.9.1 that $\mathbf{X}'\mathbf{\Lambda}^{-1}\mathbf{X}$ has a $\chi^2(2)$ distribution.

3. Let $\mathbf{a} = (a_1, a_2)'$ so that

$$X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = X_2 - \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = X_2 - a_1 X_1 - a_2 X_3.$$

If we let $\mathbf{Y} = (X_2, X_2 - a_1X_1 - a_2X_3)'$ then by Theorem V.3.1, $\mathbf{Y}$ has a multivariate normal distribution since its components are linear combinations of $\mathbf{X}$, a multivariate normal. Thus, by Theorem V.7.1, we know the components of $\mathbf{Y}$ are independent iff they are uncorrelated. Since

$$\operatorname{cov}(X_2, X_2 - a_1X_1 - a_2X_3) = \operatorname{var}(X_2) - a_1 \operatorname{cov}(X_2, X_1) - a_2 \operatorname{cov}(X_2, X_3) = 3 - a_1 - 2a_2$$

we must choose a_1 and a_2 such that $3 - a_1 - 2a_2 = 0$. For example, choosing $\mathbf{a} = (1, 1)'$ or $\mathbf{a} = (3, 0)'$ work (along with infinitely many other possibilities).

4. (a) The joint density of $(X_{(1)}, X_{(2)})'$ is

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = 2a^2\theta^{-2a} y_1^{a-1} y_2^{a-1}$$

provided that $0 < y_1 < y_2 < \theta$. If we now let $U = X_{(1)}/X_{(2)}$ and $V = X_{(2)}$, then solving for $X_{(1)}$ and $X_{(2)}$ gives $X_{(1)} = UV$ and $X_{(2)} = V$ so that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

By Theorem I.2.1, the joint density of $(U, V)'$ is therefore given by

$$f_{U,V}(u, v) = f_{X_{(1)},X_{(2)}}(uv, v) \cdot |J| = v \cdot 2a^2\theta^{-2a}(uv)^{a-1}v^{a-1} = 2a^2\theta^{-2a}u^{a-1}v^{2a-1}$$

provided that $0 < u < 1$ and $0 < v < \theta$. Hence,

$$\begin{aligned} f_U(u) = \int_0^\theta 2a^2\theta^{-2a}u^{a-1}v^{2a-1}\,\mathrm{d}v &= a\theta^{-2a}u^{a-1} \int_0^\theta 2av^{2a-1}\,\mathrm{d}v \\ &= a\theta^{-2a}u^{a-1}v^{2a}\Big|_{v=0}^{v=\theta} \\ &= au^{a-1} \end{aligned}$$

for $0 < u < 1$.

4. (b) Since $f_{U,V}(u, v)$ can be written as a product of a function of u only and a function of v only, namely $f_{U,V}(u, v) = f_U(u) \cdot f_V(v)$ where $f_U(u) = au^{a-1}$, $0 < u < 1$, and $f_V(v) = 2a\theta^{-2a}v^{2a-1}$, $0 < v < \theta$, we conclude that the random variables U and V must be independent.

4. (c) We can write $E(X_{(1)})$ as

$$E(X_{(1)}) = E\left(\frac{X_{(1)}}{X_{(2)}} \cdot X_{(2)}\right) = E(U \cdot X_{(2)}) = E(U) \cdot E(X_{(2)}) = E\left(\frac{X_{(1)}}{X_{(2)}}\right) \cdot E(X_{(2)})$$

using the fact that U and $X_{(2)}$ are independent. Dividing both sides by $E(X_{(2)})$ gives the result.