Statistics 351 Fall 2008 Midterm #1 – Solutions

1. (a) By definition,

$$f_Y(y) = \int_y^\infty e^{-x}\,\mathrm{d}x = \left(-e^{-x}\right)\Big|_y^\infty = e^{-y}, \quad y > 0.$$

Note that $Y \in \text{Exp}(1)$ so that $\mathbb{E}(Y) = 1$.

1. (b) By definition,

$$f_X(x) = \int_0^x e^{-x}\,\mathrm{d}y = xe^{-x}, \quad x > 0.$$

Note that $X \in \Gamma(2,1)$.

1. (c) By definition,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-x}}{xe^{-x}} = \frac{1}{x}, \quad 0 < y < x.$$

Note that $Y|X = x \in U(0,x)$.

1. (d) By definition,

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X=x}(y)\,\mathrm{d}y = \int_0^x y \cdot \frac{1}{x}\,\mathrm{d}y = \frac{x}{2}.$$

Thus, we conclude $\mathbb{E}(Y|X) = \frac{X}{2}$.

1. (e) Using **(d)** we find

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}\left(\frac{X}{2}\right) = \frac{1}{2}\mathbb{E}(X).$$

However, from **(a)** we know that $\mathbb{E}(Y) = 1$ and so

$$\mathbb{E}(X) = 2.$$

1. (f) If $a < 1$, then we find

$$\begin{aligned} P\{Y < aX\} = \iint_{\{y<ax\}} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^{ax} e^{-x}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty axe^{-x}\,\mathrm{d}x \\ &= a\Gamma(2) \\ &= a. \end{aligned}$$

1. (g) If $U = X + Y$ and $V = 2Y$, then solving for X and Y gives

$$X = U - \frac{V}{2} \quad \text{and} \quad Y = \frac{V}{2}.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1/2 \\ 0 & 1/2 \end{vmatrix} = 1/2.$$

Therefore, we conclude

$$f_{U,V}(u,v) = f_{X,Y}(u - v/2, v/2) \cdot |J| = e^{-u+v/2} \cdot \frac{1}{2} = \frac{1}{2} e^{-u+v/2}$$

provided that $0 < v < u < \infty$. The marginal for U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \frac{1}{2} e^{-u} \int_0^u e^{v/2}\,\mathrm{d}v = \frac{1}{2} e^{-u} \left(2e^{v/2}\right)\Big|_0^u = e^{-u}(e^{u/2} - 1) = e^{-u/2} - e^{-u}$$

provided $u > 0$.

2. (a) Notice that

$$X_{n+1} = \left(\frac{1-p}{p}\right)^{S_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n + Y_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n} \left(\frac{1-p}{p}\right)^{Y_{n+1}}$$

Therefore,

$$\begin{aligned} \mathbb{E}(X_{n+1}|X_0, \ldots, X_n) &= \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{S_n} \left(\frac{1-p}{p}\right)^{Y_{n+1}} |X_0, \ldots, X_n\right) \\ &= \left(\frac{1-p}{p}\right)^{S_n} \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}} |X_0, \ldots, X_n\right) \\ &= \left(\frac{1-p}{p}\right)^{S_n} \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right) \end{aligned}$$

where the second equality follows from "taking out what is known" and the third equality follows from the fact that $Y_1, Y_2, \ldots$ are independent. We now compute

$$\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right) = p\left(\frac{1-p}{p}\right)^1 + (1-p)\left(\frac{1-p}{p}\right)^{-1} = (1-p) + p = 1$$

and so we conclude

$$\mathbb{E}(X_{n+1}|X_0, \ldots, X_n) = \left(\frac{1-p}{p}\right)^{S_n} = X_n.$$

Hence, $\{X_n, n = 0, 1, 2, \ldots\}$ is, in fact, a martingale

2. (b) Since X_n is a martingale, we know that $\mathbb{E}(X_{n+1}|X_n) = X_n$. Therefore, using properties of conditional expectation we find

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|X_n)) = \mathbb{E}(X_n) = \cdots = \mathbb{E}(X_0).$$

Since $S_0 = 0$ we see that $X_0 = 1$ so that $\mathbb{E}(X_0) = 1$ and therefore $E(X_n) = 1$ for all $n = 0, 1, 2, \ldots$.

3. If $U = Y - X$ and $V = X$, then solving for X and Y gives

$$X = V \quad \text{and} \quad Y = U + V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

Therefore, we conclude

$$f_{U,V}(u,v) = f_{X,Y}(v, u+v) \cdot |J| = c(c+1)(b-a)^c u^{-c-2} \cdot 1 = c(c+1)(b-a)^c u^{-c-2}$$

provided that $-\infty < v < a$ and $-\infty < b - v < u < \infty$.

You should check that $\displaystyle\int_{-\infty}^{a} \int_{b-v}^{\infty} c(c+1)(b-a)^c u^{-c-2} \, \mathrm{d}u \, \mathrm{d}v = 1.$

4. By the law of total probability,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y=y}(x) f_Y(y) \, \mathrm{d}y = \int_0^{\infty} \frac{y^q}{\Gamma(q)} x^{q-1} e^{-yx} \cdot \frac{a^p}{\Gamma(p)} y^{p-1} e^{-ay} \, \mathrm{d}y$$
$$= \frac{a^p}{\Gamma(p)\Gamma(q)} x^{q-1} \int_0^{\infty} y^{p+q-1} e^{-y(x+a)} \, \mathrm{d}y.$$

We now recognize

$$\int_0^{\infty} y^{p+q-1} e^{-y(x+a)} \, \mathrm{d}y$$

as a gamma function. That is, make the change of variables $u = y(x+a)$ so that $\mathrm{d}u = (x+a) \, \mathrm{d}y$ and

$$\int_0^{\infty} y^{p+q-1} e^{-y(x+a)} \, \mathrm{d}y = \int_0^{\infty} \left(\frac{u}{x+a}\right)^{p+q-1} e^{-u} \frac{\mathrm{d}u}{x+a} = \left(\frac{1}{x+a}\right)^{p+q} \int_0^{\infty} u^{p+q-1} e^{-u} \, \mathrm{d}u$$
$$= \frac{\Gamma(p+q)}{(x+a)^{p+q}}.$$

Hence, we conclude that

$$f_X(x) = \frac{a^p}{\Gamma(p)\Gamma(q)} x^{q-1} \cdot \frac{\Gamma(p+q)}{(x+a)^{p+q}} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{a^p x^{q-1}}{(x+a)^{p+q}}, \quad 0 < x < \infty.$$

This is sometimes called a type II generalized Pareto density with parameters q, p, a.