Statistics 351 Fall 2007 Midterm #1 – Solutions

1. (a) By definition,

$$f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy = 2e^{-x}\left(-e^{-y}\right)\Big|_x^\infty = 2e^{-2x}, \quad x > 0,$$

and

$$f_Y(y) = \int_0^y 2e^{-x}e^{-y}\,dx = 2e^{-y}\left(-e^{-x}\right)\Big|_0^y = 2e^{-y}\left(1 - e^{-y}\right), \quad y > 0.$$

1. (b) Since $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$, we immediately conclude that X and Y are not independent random variables.

1. (c) By definition,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x-y}}{2e^{-2x}} = e^{x-y}, \quad 0 < x < y < \infty.$$

1. (d) By definition,

$$E(Y|X=x) = \int_{-\infty}^\infty y \cdot f_{Y|X=x}(y)\,dy = \int_x^\infty y \cdot e^{x-y}\,dy.$$

Let $u = y - x$ so that $du = dy$ and the integral above becomes

$$\int_x^\infty y \cdot e^{x-y}\,dy = \int_0^\infty (u+x)e^{-u}\,du = \int_0^\infty ue^{-u}\,du + x\int_0^\infty e^{-u}\,du = \Gamma(2) + x\Gamma(1) = 1 + x$$

and so $\mathbb{E}(Y|X) = 1 + X$.

1. (e) Using **(d)** we find $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(1+X) = 1 + \mathbb{E}(X)$. However,

$$\mathbb{E}(X) = \int_{-\infty}^\infty x \cdot f_X(x)\,dx = \int_0^\infty 2xe^{-2x}\,dx.$$

Let $u = 2x$ so that $du = 2dx$ and the integral above becomes

$$\int_0^\infty 2xe^{-2x}\,dx = \frac{1}{2}\int_0^\infty ue^{-u}\,du = \frac{1}{2} \cdot \Gamma(2) = \frac{1}{2}.$$

Therefore,

$$\mathbb{E}(Y) = 1 + \mathbb{E}(X) = 1 + \frac{1}{2} = \frac{3}{2}.$$

1. (f) If $U = X + Y$ and $V = X$, then solving for X and Y gives

$$X = V \quad \text{and} \quad Y = U - V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

Therefore, we conclude

$$f_{U,V}(u, v) = f_{X,Y}(v, u - v) \cdot |J| = 2e^{-v-(u-v)} \cdot 1 = 2e^{-u}$$

provided that $0 < 2v < u < \infty$ (or, equivalently, $0 < v < \frac{u}{2} < \infty$). The marginal for U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv = \int_0^{u/2} 2e^{-u}\, dv = \frac{u}{2} \cdot 2e^{-u} = ue^{-u}, \quad u > 0.$$

We recognize this as the density of a $\Gamma(2, 1)$ random variable. That is, $U = X + Y \in \Gamma(2, 1)$ as required.

2. (a) We find

$$\begin{aligned} \mathbb{E}(X_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}(X_n \cdot Y_{n+1}|X_1, \ldots, X_n) \\ &= X_n \mathbb{E}(Y_{n+1}|X_1, \ldots, X_n) \quad \text{(by taking out what is known)} \\ &= X_n \mathbb{E}(Y_{n+1}) \quad \text{(since } Y_{n+1} \text{ is independent of } X_1, \ldots, X_n) \\ &= X_n \cdot 1 \\ &= X_n \end{aligned}$$

and so $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

2. (b) For $n = 1, 2, \ldots$, we find

$$\mathbb{E}(X_n) = \mathbb{E}(Y_1 \cdot Y_2 \cdots Y_n) = \mathbb{E}(Y_1) \cdot \mathbb{E}(Y_2) \cdots \mathbb{E}(Y_n) = 1$$

using the fact that $Y_1, Y_2, \ldots$ are independent.

3. (a) By the law of total probability,

$$P(X = 0) = \int_0^1 P(X = 0|A = a) f_A(a)\, da = \int_0^1 (1 - a)\, da = a - \frac{a^2}{2}\Big|_0^1 = \frac{1}{2}$$

and

$$P(X = 1) = \int_0^1 P(X = 1|A = a) f_A(a)\, da = \int_0^1 a\, da = \frac{a^2}{2}\Big|_0^1 = \frac{1}{2}.$$

3. (b) By definition,

$$f_{A|X=0}(a) = \frac{P(X=0|A=a)f_A(a)}{P(X=0)} = \frac{(1-a)\cdot 1}{1/2} = 2(1-a), \quad 0<a<1,$$

and

$$f_{A|X=1}(a) = \frac{P(X=1|A=a)f_A(a)}{P(X=1)} = \frac{a\cdot 1}{1/2} = 2a, \quad 0<a<1.$$

4. If $U = \sqrt{3}\,X + Y$ and $V = X - \sqrt{3}\,Y$, then solving for X and Y gives

$$X = \frac{\sqrt{3}\,U + V}{4} \quad \text{and} \quad Y = \frac{U - \sqrt{3}\,V}{4}.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sqrt{3}/4 & 1/4 \\ 1/4 & -\sqrt{3}/4 \end{vmatrix} = -1/4.$$

Therefore, we conclude

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\left(\frac{\sqrt{3}\,u+v}{4}, \frac{u-\sqrt{3}\,v}{4}\right)\cdot |J| \\ &= \frac{1}{4}\cdot f_X\left(\frac{\sqrt{3}\,u+v}{4}\right)\cdot f_Y\left(\frac{u-\sqrt{3}\,v}{4}\right), \quad -\infty<u,v<\infty, \end{aligned}$$

using the assumed independence of X and Y. The exact form of f_X and f_Y gives

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{4}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\sqrt{3}\,u+v}{4}\right)^2\right\}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-\sqrt{3}\,v}{4}\right)^2\right\} \\ &= \frac{1}{8\pi}\exp\left\{-\frac{1}{2\cdot 16}\left(3u^2+2\sqrt{3}\,uv+v^2+u^2-2\sqrt{3}\,uv+3v^2\right)\right\} \\ &= \frac{1}{8\pi}\exp\left\{-\frac{1}{2\cdot 16}\left(4u^2+4v^2\right)\right\} \\ &= \frac{1}{2\sqrt{2\pi}}\exp\left\{-\frac{u^2}{2\cdot 4}\right\}\cdot\frac{1}{2\sqrt{2\pi}}\exp\left\{-\frac{u^2}{2\cdot 4}\right\} \end{aligned}$$

provided that $-\infty<u<\infty$, $-\infty<v<\infty$. Hence, we immediately conclude that $f_{U,V}(u,v) = f_U(u)\cdot f_V(v)$ and so U and V are independent random variables. Furthermore, we recognize that both U and V have a $N(0,4)$ density. Together, this implies that $U|V=v \in N(0,4)$.