Statistics 351 Fall 2006 Midterm #2 — Solutions

1. (a) Recall that a square matrix is strictly positive definite if and only if the determinants of all of its upper block diagonal matrices are strictly positive. Since

$$\mathbf{\Lambda} = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$$

we see that $\det(\mathbf{\Lambda}_1) = \det(2) = 2 > 0$ and $\det(\mathbf{\Lambda}_2) = \det(\mathbf{\Lambda}) = 6 - 4 = 2 > 0$, and so we conclude that $\mathbf{\Lambda}$ is, in fact, strictly positive definite.

1. (b) Since $\det[\mathbf{\Lambda}] = 2$, we find

$$\mathbf{\Lambda}^{-1} = \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix}$$

so that

$$(\mathbf{x} - \boldsymbol{\mu})'\mathbf{\Lambda}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = \frac{3}{2}(x_1 - 1)^2 + 2(x_1 - 1)(x_2 - 2) + (x_2 - 2)^2.$$

Thus, the density of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{2}} \exp\left\{ -\frac{1}{2}\left(\frac{3}{2}(x_1 - 1)^2 + 2(x_1 - 1)(x_2 - 2) + (x_2 - 2)^2 \right) \right\}.$$

1. (c) If $Y_1 = X_1 - 2X_2$ and $Y_2 = X_1 + X_2$, then by Definition I, Y_1 is normal with mean $E(Y_1) = E(X_1) - 2E(X_2) = -3$ and variance $\operatorname{var}(Y_1) = \operatorname{var}(X_1) + 4\operatorname{var}(X_2) - 4\operatorname{cov}(X_1, X_2) = 22$, and Y_2 is normal with mean $E(Y_2) = E(X_1) + E(X_2) = 3$ and variance $\operatorname{var}(Y_1) = \operatorname{var}(X_1) + \operatorname{var}(X_2) - 2\operatorname{cov}(X_1, X_2) = 1$. Since

$$\operatorname{cov}(Y_1, Y_2) = \operatorname{cov}(X_1 - 2X_2, X_1 + X_2) = \operatorname{var}(X_1) - \operatorname{cov}(X_1, X_2) - 2\operatorname{var}(X_2) = -2,$$

we conclude

$$\mathbf{Y} = (Y_1, Y_2)' \in N\left(\begin{pmatrix} -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 22 & -2 \\ -2 & 1 \end{pmatrix} \right).$$

2. By Definition I, we see that X_1 and $X_1 + X_2$ are each normally distributed random variables. Therefore, by Theorem V.7.1, X_1 and $X_1 + X_2$ are independent if and only if they are uncorrelated. Since,

$$\operatorname{cov}(X_1, X_1 + X_2) = \operatorname{cov}(X_1, X_1) + \operatorname{cov}(X_1, X_2) = \operatorname{var}(X_1) + \operatorname{cov}(X_1, X_2) = 1 - 1 = 0,$$

we conclude that X_1 and $X_1 + X_2$ are, in fact, independent.

3. The joint density of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})'$ is given by

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4) = 4!, \quad 0 < y_1 < y_2 < y_3 < y_4 < 1.$$

The joint density of $(X_{(2)}, X_{(3)})'$ is then given by

$$f_{X_{(2)},X_{(3)}}(y_2, y_3) = \int_{y_3}^{1} \int_{0}^{y_2} 24\, dy_1\, dy_4 = 24y_2(1 - y_3), \quad 0 < y_2 < y_3 < 1.$$

Let $U = X_{(3)} - X_{(2)}$ and $V = X_{(2)}$ so that $X_{(2)} = V$ and $X_{(3)} = U + V$. The Jacobian of this transformation is

$$J = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

so by the transformation theorem (Theorem I.2.1) the joint density of U and V is

$$f_{U,V}(u, v) = 24v(1 - u - v)$$

provided that $0 < v < 1 - u$ and $0 < u < 1$. Thus, the marginal for U is

$$f_U(u) = \int_0^{1-u} 24v(1 - u - v)\, dv = (12v^2(1 - u) - 8v^3)\Big|_0^{(1-u)} = 4(1 - u)^3, \quad 0 < u < 1.$$

That is, the density for $X_{(3)} - X_{(2)}$ is

$$f_{X_{(3)}-X_{(2)}}(u) = 4(1 - u)^3, \quad 0 < u < 1.$$

4. Let $Y = \max\{X_1, X_2\}$. By Theorem IV.1.2, the density function for Y is given by

$$f_Y(y) = 2F(y)f(y)$$

where F is the common distribution function of X_1 and X_2, and f is their common density function. (Recall that X_1 and X_2 are iid $N(0, 1)$.) Therefore,

$$\begin{aligned} E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy &= 2\int_{-\infty}^{\infty} yF(y)f(y)\, dy \\ &= 2\int_{-\infty}^{\infty} y \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{y} y e^{-\frac{y^2}{2}} e^{-\frac{x^2}{2}}\, dx\, dy. \end{aligned}$$

In order to calculate this integral, we switch the order of integration so that

$$\begin{aligned} E(Y) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{x}^{\infty} y e^{-\frac{y^2}{2}} e^{-\frac{x^2}{2}}\, dy\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left[\int_x^{\infty} y e^{-\frac{y^2}{2}}\, dy\right] dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left[-e^{-\frac{y^2}{2}}\right]_{y=x}^{y=\infty} dx = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{1}{\pi}\cdot\sqrt{\pi} = \frac{1}{\sqrt{\pi}}. \end{aligned}$$

The last line follows from the fact that the density function of a $N(0, 1/2)$ random variable integrates to 1.

5. By Definition I, we see that $X_1 - \rho X_2$ is normally distributed with mean

$$E(X_1 - \rho X_2) = E(X_1) - \rho E(X_2) = 0$$

and variance

$$\operatorname{var}(X_1 - \rho X_2) = \operatorname{var}(X_1) + \rho^2 \operatorname{var}(X_2) - 2\rho \operatorname{cov}(X_1, X_2) = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2.$$

That is, $X_1 - \rho X_2 = Y$ where $Y \in N(0, 1 - \rho^2)$. Hence, $Y = \sqrt{1-\rho^2}Z$ where $Z \in N(0,1)$. In other words, there exists a $Z \in N(0,1)$ such that

$$X_1 - \rho X_2 = \sqrt{1-\rho^2}Z.$$