Lecture #24: Conditional Distributions for the Bivariate Normal

Reference. §5.6 pages 127–130

Recall. Suppose that $\mathbf{X} = (X, Y)' \in \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}.$$

In particular, $X \in \mathcal{N}(\mu_x, \sigma_x^2)$, $Y \in \mathcal{N}(\mu_y, \sigma_y^2)$, and $\text{Corr}(X, Y) = \rho$. Assuming that $\det[\boldsymbol{\Lambda}] > 0$, or equivalently, that $\sigma_x \neq 0$, $\sigma_y \neq 0$, and $-1 < \rho < 1$, then the conditional distribution of Y given $X = x$ is

$$Y|X = x \in \mathcal{N}\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right).$$

In particular,

$$\mathbb{E}(Y|X) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)$$

and

$$\text{Var}(Y|X) = \sigma_y^2(1 - \rho^2).$$

Example. Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has density

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x_1^2 - 2x_1x_2 + 2x_2^2)\right\}.$$

(a) Determine the distribution of $\mathbf{X}$.

(b) Determine the distribution of X_2 given $X_1 = x$.

(c) Determine the distribution of $X_1 - X_2$ given $X_1 + X_2 = 0$.

Solution. (a) It appears that $\mathbf{X}$ has the density function of a multivariate normal random vector. Thus, we must determine the mean vector $\boldsymbol{\mu}$, the covariance matrix $\boldsymbol{\Lambda}$, and verify that the distribution is, in fact, MVN. Clearly $\boldsymbol{\mu} = \overline{0}$. Furthermore,

$$x_1^2 - 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which implies that

$$\boldsymbol{\Lambda}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

It is easy to invert a 2×2 matrix, and so we see immediately that

$$\boldsymbol{\Lambda} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

since $\det[\boldsymbol{\Lambda}] = 1$.

Thus, matching the general form of a MVN density function with the function given in this problem, we conclude that

$$\mathbf{X} \in \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\right).$$

(b) The conditional density of X_2 given $X_1 = x$ is

$$f_{X_2|X_1=x}(x_2) = \frac{f_{X_1,X_2}(x, x_2)}{f_{X_1}(x)} = \frac{\frac{1}{2\pi}\exp\left\{-\frac{1}{2}(x^2 - 2xx_2 + 2x_2^2)\right\}}{\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2}}\exp\left\{-\frac{1}{2}\cdot\frac{x^2}{2}\right\}}$$

since $X_1 \in N(0, 2)$. Therefore,

$$f_{X_2|X=x}(x_2) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{1/2}}\exp\left\{-\frac{1}{2}\frac{(x_2 - x/2)^2}{1/2}\right\}$$

so that $X_2|X = x \in N\left(\frac{x}{2}, \frac{1}{2}\right)$.

(c) Consider $\mathbf{Y} = (Y_1, Y_2)'$ where $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. If we let

$$B = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

so that

$$B\mathbf{X} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}X_1\\X_2\end{bmatrix} = \begin{bmatrix}X_1 - X_2\\X_1 + X_2\end{bmatrix} = \begin{bmatrix}Y_1\\Y_2\end{bmatrix} = \mathbf{Y},$$

then by Theorem 5.3.1, we conclude that $\mathbf{Y} \in \mathcal{N}(B\boldsymbol{\mu}, B\boldsymbol{\Lambda}B')$ where

$$\mathbb{E}(\mathbf{Y}) = B\boldsymbol{\mu} = \begin{bmatrix}0\\0\end{bmatrix}$$

and

$$\text{Cov}(\mathbf{Y}) = B\boldsymbol{\Lambda}B' = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 5\end{bmatrix}.$$

Since $\det[\text{Cov}(\mathbf{Y})] = 4$, we deduce that the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(y_1, y_2) = \frac{1}{4\pi}\exp\left\{-\frac{5}{8}\left(y_1^2 - \frac{2}{5}y_1y_2 + \frac{1}{5}y_2^2\right)\right\}.$$

We now find

$$f_{Y_1|Y_2=0}(y_1) = \frac{f_{\mathbf{Y}}(y_1, 0)}{f_{Y_2}(0)} = \frac{\frac{1}{4\pi}\exp\left\{-\frac{5}{8}\left(y_1^2 - \frac{2}{5}y_1(0) + \frac{1}{5}(0)^2\right)\right\}}{\frac{1}{\sqrt{10\pi}}\exp\left\{-\frac{1}{10}(0)\right\}}$$

$$= \frac{\sqrt{5}}{2\sqrt{2\pi}}\exp\left\{-\frac{5}{8}y_1^2\right\}.$$

In other words, the distribution of $Y_1|Y_2 = 0$, or equivalently, $X_1 - X_2|X_1 + X_2 = 0$, is $\mathcal{N}(0, \frac{4}{5})$.