Statistics 351 (Fall 2015) October 7, 2015
Prof. Michael Kozdron

Lecture #13: Conditional Expectation

Reference. §2.2 pages 33–37

Example. Suppose that X is the time before the first occurrence of a radioactive decay which is measured by an instrument. However, there is a delay built into the mechanism, and the decay is recorded as having taken place at some time $Y > X$. (That is, although we observe $Y = y$ as our time of decay, the first occurrence of the decay actually takes place at $X = x$. Because of the delay in the mechanism we *know* that $y > x$.) Assume that the prior distribution for X is Exp(1) and that the conditional distribution of Y given $X = x$ is

$$f_{Y|X=x}(y) = \lambda e^{-\lambda(y-x)}, \quad 0 < x < y < \infty,$$

where $\lambda > 0$ is a known constant. Determine the posterior density function $f_{X|Y=y}(x)$.

Solution. By definition,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X=x}(y) f_X(x)}{f_Y(y)}.$$

We are told that

$$f_{Y|X=x}(y) = \lambda e^{-\lambda(y-x)}, \quad 0 < x < y < \infty,$$

and

$$f_X(x) = e^{-x}, \quad 0 < x < \infty.$$

We now compute

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_{-\infty}^{\infty} f_{Y|X=x}(y) f_X(x)\,\mathrm{d}x &= \lambda \int_0^y e^{-\lambda(y-x)} e^{-x}\,\mathrm{d}x \\ &= \lambda e^{-\lambda y} \int_0^y e^{-(1-\lambda)x}\,\mathrm{d}x \\ &= \lambda e^{-\lambda y} \left[-\frac{1}{1-\lambda} e^{-(1-\lambda)x} \right]_{x=0}^{x=y} \\ &= \frac{\lambda}{1-\lambda} e^{-\lambda y} \left[1 - e^{-(1-\lambda)y} \right] \end{aligned}$$

for $0 < y < \infty$, and so we conclude

$$f_{X|Y=y}(x) = \frac{\lambda e^{-\lambda(y-x)} e^{-x}}{\frac{\lambda}{1-\lambda} e^{-\lambda y} \left[1 - e^{-(1-\lambda)y}\right]} = \frac{(1-\lambda) e^{(\lambda-1)x}}{1 - e^{(\lambda-1)y}}$$

for $0 < x < y$.

Example. A stick of length 1 is broken at a random point (uniformly over the stick). The remaining piece is broken once more. Find the expected value and variance of the length of the remaining piece.

Solution. Let $X \in U(0,1)$ denote the position of the first random break, and let $Y \in U(0,X)$ denote the position of the second random break.

The interpretation of the distribution of Y is that given $X = x$, the random variable $Y \in U(0,x)$, i.e., $Y|X = x \in U(0,x)$ with $X \in U(0,1)$, so that

$$f_{Y|X=x}(y) = \begin{cases} 1/x, & \text{for } 0 < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

Intuitively,

$$\mathbb{E}(Y|X = x) = \frac{x}{2} \quad \text{and} \quad \operatorname{Var}(Y|X = x) = \frac{x^2}{12}.$$

Formally, we have the following definition.

Definition. If $(X,Y)'$ is jointly distributed, then the *conditional expectation of Y given $X = x$* is

$$E(Y|X = x) = \begin{cases} \displaystyle\sum_y y\, p_{Y|X=x}(y), & \text{in the discrete case, and} \\ \displaystyle\int_{-\infty}^{\infty} y\, f_{Y|X=x}(y)\,\mathrm{d}y, & \text{in the continuous case.} \end{cases}$$

provided that the sum or integral converges absolutely.

Example (continued). We can now verify that $\mathbb{E}(Y|X = x) = x/2$. This follows since

$$\mathbb{E}(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)\,\mathrm{d}y = \int_0^x y \cdot \frac{1}{x} dy = \frac{x}{2}$$

as expected.

Note. The conditional expectation of Y given $X = x$ depends on the value of x. That is, $\mathbb{E}(Y|X = x)$ is a function of x, say

$$\mathbb{E}(Y|X = x) = h(x).$$