Statistics 351 (Fall 2015) October 5, 2015
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Lecture #12: The Bayesian Approach

The basic goal of statistics is to estimate population parameters. Data is collected and used to form the required estimator. The point-of-view of *frequentist statistics* is that parameter estimation is based on "long-run averages" which is justified by results such as the central limit theorem and the strong law of large numbers.

However, another point-of-view is that the experimenter never has complete lack of knowledge of the population parameter, but rather has some prior (*a priori*) knowledge or a "reasonable guess" of what the parameter is. The data are then collected and used to update (*a posteriori*) information about the parameter. This second approach is known as *Bayesian statistics*.

Example. Consider tossing a coin but assume that nothing is known about $q = P\{\text{head}\}$. Let X_n denote the number of heads after n tosses. One possible model for this situation is

$$X_n|Q = q \in \text{Bin}(n, q) \text{ with } Q \in U(0, 1).$$

(a) Determine the unconditional distribution of X_n.

(b) Determine the posterior distribution of Q given $X_n = k$.

Solution. (a) For $k = 0, 1, 2, \dots, n$, we obtain from the law of total probability that

$$\begin{aligned}
P\{X_n = k\} &= \int_0^1 P\{X_n = k|Q = q\} f_Q(q)\,\mathrm{d}q \\
&= \int_0^1 \binom{n}{k} q^k (1-q)^{n-k} \cdot 1\,\mathrm{d}q \\
&= \binom{n}{k} \int_0^1 q^{(k+1)-1}(1-q)^{(n+1-k)-1}\,\mathrm{d}q \\
&= \binom{n}{k} \frac{\Gamma(k+1)\Gamma(n+1-k)}{\Gamma(k+1+n+1-k)} \quad \text{using facts about the beta distribution} \\
&= \frac{n!k!(n-k)!}{(n-k)!k!(n+1)!} \\
&= \frac{1}{n+1}.
\end{aligned}$$

That is, X_n is uniform on $\{0, 1, \dots, n\}$.

(b) As for the posterior distribution of Q given $X_n = k$, we find using Bayes' Rule (i.e., the definition of conditional probability) that

$$\begin{aligned} f_{Q|X_n=k}(q) = \frac{P\{X_n = k|Q = y\} f_Q(y)}{P\{X_n = k\}} &= \frac{\binom{n}{k} q^k (1-q)^{n-k} \cdot 1}{\frac{1}{n+1}} \\ &= (n+1)\binom{n}{k} q^k (1-q)^{n-k} \\ &= (n+1)\frac{n!}{k!(n-k)!} q^k (1-q)^{n-k} \\ &= \frac{(n+1!}{k!(n-k)!} q^k (1-q)^{n-k} \\ &= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n+1-k)} q^k (1-q)^{n-k} \end{aligned}$$

provided that $0 < q < 1$. Note that we used the fact that $\Gamma(p+1) = p!$ whenever p is a positive integer. In summary, the posterior distribution of Q given $X_n = k$ is $\beta(k+1, n+1-k)$.

Example. If $X|N = n \in \text{Bin}(n,p)$ with $N \in \text{Po}(\lambda)$, determine the distribution of X.

Solution. If $X|N = n \in \text{Bin}(n,p)$, then for $k = 0, 1, 2, \ldots$,

$$P\{X = k|N = n\} = \binom{n}{k} p^k (1-p)^{n-k}.$$

Therefore, if $k = 0, 1, 2, \ldots$, it follows that

$$P\{X = k\} = \sum_{n=0}^{\infty} P\{X = k|N = n\} P\{N = n\} = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{e^{-\lambda}\lambda^n}{n!}$$

Note that we used the fact that n must necessarily be at least equal to k. Indeed, if we have observed k successes, then there necessarily must have been *at least* k trials. Continuing, we find

$$P\{X = k\} = \frac{p^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{n!(1-p)^{n-k}\lambda^n}{(n-k)!n!} = \frac{p^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} \lambda^n.$$

We now observe that two of the terms in the summand depend on n only through $n-k$. Moreover, we see that if $n = k$, then $n - k = 0$. Thus means that we want to write $\lambda^n = \lambda^{n-k+k} = \lambda^k \lambda^{n-k}$ which implies that

$$\begin{aligned} P\{X = k\} = \frac{p^k \lambda^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} \lambda^{n-k} &= \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} \\ &= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} \\ &= \frac{(\lambda p)^k e^{-\lambda}}{k!} \cdot e^{\lambda(1-p)} \\ &= \frac{(\lambda p)^k e^{-\lambda p}}{k!}. \end{aligned}$$

Thus, we conclude that $X \in \text{Po}(\lambda p)$.