Statistics 351 (Fall 2015) September 16, 2015
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Lecture #4: Functions of Random Variables

Reference. §1.2 pages 19–24

Example. Let $X \in \text{Exp}(1/\lambda_1)$ and $Y \in \text{Exp}(1/\lambda_2)$ be independent. Compute $P\{X < Y\}$.

Solution. Since X and Y are independent, we know that the joint density $f_{X,Y}(x,y)$ is simply the product of the marginal densities $f_X(x)$ and $f_Y(y)$. That is,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2 y} = \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}$$

provided that $x > 0$ and $y > 0$. Recall that in the one-dimensional case, if we want to compute the probability of an event such as $P\{a < X < b\}$ we simply need to integrate the density function for X over the appropriate region. That is,

$$P\{a < X < b\} = \int_a^b f_X(x)\,\mathrm{d}x = \int_a^b \lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x = e^{-a\lambda_1} - e^{-b\lambda_1}.$$

In the multi-dimensional case, we do the same thing. That is, we integrate the joint density over the appropriate region:

$$P\{X < Y\} = \iint_{\{x<y\}} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda_1\lambda_2 \iint_{\{x<y\}} e^{-\lambda_1 x - \lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y.$$

There are two equivalent ways to compute this iterated integral. The first is

$$\begin{aligned}
\lambda_1\lambda_2 \iint_{\{x<y\}} e^{-\lambda_1 x - \lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y &= \lambda_1\lambda_2 \int_0^\infty \int_0^y e^{-\lambda_1 x - \lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 y} \left[\int_0^y \lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x\right] \mathrm{d}y \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 y} \left[1 - e^{-\lambda_1 y}\right] \mathrm{d}y \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 y}\,\mathrm{d}y - \lambda_2 \int_0^\infty e^{-y(\lambda_1+\lambda_2)}\,\mathrm{d}y \\
&= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

The second is

$$\begin{aligned}
\lambda_1\lambda_2 \iint_{\{x<y\}} e^{-\lambda_1 x - \lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y = \lambda_1\lambda_2 \int_0^\infty \int_x^\infty e^{-\lambda_1 x - \lambda_2 y}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty \lambda_1 e^{-\lambda_1 x} \left[\int_x^\infty \lambda_2 e^{-\lambda_2 y}\,\mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 x}\,\mathrm{d}x \\
&= \int_0^\infty \lambda_1 e^{-x(\lambda_1+\lambda_2)}\,\mathrm{d}x \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

In either case, we find

$$P\{X < Y\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Notice that if $\lambda_1 = \lambda_2$, then $P\{X < Y\} = 1/2$ as expected. (Why?)

Motivation. Sometimes we are interested in the distribution of a function of a random variable.

Stat 252 Motivation. The pivot method for constructing confidence intervals required one to do exactly that. We did, however, restrict ourselves to strictly increasing functions of one dimensional random variables.

Example. Suppose that $X \in \text{Exp}(\lambda)$. Determine the distribution/density of $Y = e^X$.

Solution. If $X \in \text{Exp}(\lambda)$, then

$$F_X(x) = 1 - e^{-x/\lambda}, \quad x > 0, \quad \text{and so} \quad f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0.$$

Therefore,

$$\begin{aligned} F_Y(y) = P\{Y \le y\} = P\{e^X \le y\} &= P\{X \le \log y\} \\ &= \int_0^{\log y} \frac{1}{\lambda} e^{-x/\lambda} \, \mathrm{d}x \\ &= -e^{-x/\lambda} \Big|_0^{\log y} \\ &= 1 - e^{-\log y/\lambda} \\ &= 1 - y^{-1/\lambda}, \quad y > 1. \end{aligned}$$

We now find $f_Y(y)$.

- Method #1:

$$\begin{aligned} f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}(1 - y^{-1/\lambda}) \\ &= \frac{1}{\lambda} y^{-1-1/\lambda}. \end{aligned}$$

- Method #2:

$$\begin{aligned} f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\log y} \frac{1}{\lambda} e^{-x/\lambda} \, \mathrm{d}x \\ &= \frac{1}{\lambda} e^{-\log y/\lambda} \cdot \frac{\mathrm{d}}{\mathrm{d}y}(\log y) \quad \text{(chain rule)} \\ &= \frac{1}{\lambda} y^{-1/\lambda} \cdot \frac{1}{y} \quad \text{as above.} \end{aligned}$$

Remark. We observe that Method #2 can be generalized to any strictly increasing function g provided that its derivative g' exists.

Theorem and Proof. If X is a continuous random variables and g is a strictly increasing, differentiable function, then if $Y = g(X)$,

- $F_Y(y) = P\{Y \le y\} = P\{g(X) \le y\} = P\{X \le g^{-1}(y)\} = F_X(g^{-1}(y))$, and
- $f_Y(y) = \dfrac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \dfrac{\mathrm{d}}{\mathrm{d}y} \displaystyle\int_{-\infty}^{g^{-1}(y)} f_X(x)\,\mathrm{d}x = f_X(g^{-1}(y)) \cdot \dfrac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)$.

On the other hand, if g is strictly decreasing, then

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y).$$

(The extra minus sign is needed since $\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) < 0$.)

Summary. If g is strictly monotone, then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right|.$$

Exercise. Read Examples 2.1, 2.2 on pages 19–20. Although Gut [1] skips some steps, you can now fill them in.

Remark. When you need to change variables, don't try to just plug into a memorized formula. Instead, follow "Method #2" directly as in the example.

Problem. You can now solve Problems #1, #2, #3, #4, #6, #7 on page 24 of [1]. (Note that Problem #6 is very easy and does not require this result.)

Remark. We will examine the multi-dimensional case next class.