Statistics 351 (Fall 2015) September 9, 2015
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Lecture #1: Introduction to Multivariate Probability

Suppose that X is a *continuous random variable*. We know from Stat 251 that information about X is encoded in its distribution function

$$F_X(x) = P\{X \le x\}.$$

Saying that X is *continuous* means that there exists some function $f_X : \mathbb{R} \to [0, \infty)$ with the properties that

- $f_X(x) \ge 0$ for all $x \in \mathbb{R}$,
- $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$, and
- $\displaystyle F_X(x) = \int_{-\infty}^{x} f_X(u)\,\mathrm{d}u$.

Of course, we call f_X the (probability) *density* (function) of X.

Certain densities have special names.

Example. If

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty,$$

then X is said to have a normal distribution with mean 0 and variance 1. We write this as either $X \sim \mathcal{N}(0,1)$ or $X \in \mathcal{N}(0,1)$.

Example. Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = 1.$$

Solution. We can prove this result using polar coordinates. (Math 213 *is* a prerequisite!) Let

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x$$

so that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2/2}\,\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y.$$

We recognize that in order to evaluate this integral, we need to change to polar coordinates. Therefore, let $x = r\cos\theta$ and $y = r\sin\theta$ for $0 \le r < \infty$, $0 \le \theta < 2\pi$, so that $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ and

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta = 2\pi.$$

Thus, $I^2 = 2\pi$ from which we conclude $I = \sqrt{2\pi}$. In other words,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi} \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = 1.$$

Remark. The change-of-variables from cartesian coordinates to polar coordinates was of the form

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

That is, the old variables x and y were converted to new variables r and θ, but the conversion required the new variables to be multiplied by a function of r and θ, namely r. (In fact, this function is $|J(r, \theta)| = r$.) In this course we will learn how to change variables in general. Our old variables x and y will be converted into new variables u and v with the conversion requiring multiplication by a function of u and v, namely

$$\mathrm{d}x\,\mathrm{d}y = |J(u, v)|\,\mathrm{d}u\,\mathrm{d}v,$$

where the function $J(u, v)$ is called the *Jacobian* of the transformation.

In fact, the u-substitution from first-year calculus is a special case of this.

Example. If we want to compute

$$\int x e^{-x^2}\,\mathrm{d}x,$$

then we let $u = x^2$ so that $\mathrm{d}u = 2x\,\mathrm{d}x$. To put this in the form of a Jacobian, however, we write $x = \sqrt{u}$ so that

$$\mathrm{d}x = \frac{1}{2\sqrt{u}}\,\mathrm{d}u.$$

This gives

$$\int x e^{-x^2}\,\mathrm{d}x = \int \sqrt{u} e^{-u} \frac{1}{2\sqrt{u}}\,\mathrm{d}u = \frac{1}{2} \int e^{-u}\,\mathrm{d}u.$$