Stat 351 Fall 2015
Chapter 5 Solutions

Problem #2. Let $\mathbf{X} = (X, Y)'$ with

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right),$$

and consider the change of variables to polar coordinates $(R, \Theta)'$. The inverse of this transformation is given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

for $0 \le \theta < 2\pi$, $r > 0$ so that the Jacobian is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Since the density of $(X, Y)'$ is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}, \quad -\infty < x, y < \infty,$$

it now follows from Theorem 1.2.1 that the density of $(R, \Theta)'$ is

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r\cos\theta, r\sin\theta) \cdot |J| \\ &= r f_{X,Y}(r\cos\theta, r\sin\theta) \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(r^2\cos^2\theta - 2\rho r^2 \sin\theta\cos\theta + r^2\sin^2\theta)\right\} \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)}\right\} \end{aligned}$$

for $0 \le \theta < 2\pi$, $r > 0$. The marginal density for Θ is therefore given by

$$\begin{aligned} f_\Theta(\theta) &= \int_0^\infty \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)}\right\} \mathrm{d}r \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty r \exp\left\{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)}\right\} \mathrm{d}r. \end{aligned}$$

Making the change of variables

$$u = \frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)} \quad \text{so that} \quad \frac{(1-\rho^2)\,\mathrm{d}u}{(1-\rho\sin 2\theta)} = r\,\mathrm{d}r$$

implies that

$$f_\Theta(\theta) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \frac{(1-\rho^2)}{(1-\rho\sin 2\theta)} \int_0^\infty e^{-u}\,\mathrm{d}u = \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}$$

provided $0 \le \theta < 2\pi$.

Problem #4. If the random vector $(X,Y)'$ has a multivariate normal distribution, then it follows from Definition I that both $X+Y$ and $X-Y$ are normal random variables. If $\text{var}(X)=\text{var}(Y)$, then

$$\text{cov}(X+Y,X-Y)=\text{cov}(X,X)-\text{cov}(X,Y)+\text{cov}(Y,X)+\text{cov}(Y,Y)=\text{var}(X)-\text{var}(Y)=0.$$

Theorem 5.7.1 therefore implies that $X+Y$ and $X-Y$ are independent since $\text{cov}(X+Y,X-Y)=0$.

Problem #11. Note that by Theorem 5.7.1, in order to show X_1, X_2, and X_3 are independent, it is enough to show that $\text{cov}(X_1,X_2)=\text{cov}(X_1,X_3)=\text{cov}(X_2,X_3)=0$. Thus, if X_1 and X_2+X_3 are independent, then $\text{cov}(X_1,X_2+X_3)=\text{cov}(X_1,X_2)+\text{cov}(X_1,X_3)=0$ and so

$$\text{cov}(X_1,X_2)=-\text{cov}(X_1,X_3). \tag{1}$$

If X_2 and X_1+X_3 are independent, then $\text{cov}(X_2,X_1+X_3)=\text{cov}(X_2,X_1)+\text{cov}(X_2,X_3)=0$ and so

$$\text{cov}(X_2,X_1)=-\text{cov}(X_2,X_3). \tag{2}$$

Finally, if X_3 and X_1+X_2 are independent, then $\text{cov}(X_3,X_1+X_2)=\text{cov}(X_3,X_1)+\text{cov}(X_3,X_2)=0$ and so

$$\text{cov}(X_3,X_1)=-\text{cov}(X_3,X_2). \tag{3}$$

Since (1), (2), and (3) must be simultaneously satisfied, the only possibility is that $\text{cov}(X_1,X_2)=\text{cov}(X_1,X_3)=\text{cov}(X_2,X_3)=0$. Hence, X_1, X_2, and X_3 are independent as required.

Problem #12. Using Theorem 5.3.1, the distribution of $\mathbf{Y}=(Y_1,Y_2)'$ is

$$\mathbf{Y}\in N\left(\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}10&5\\5&5\end{pmatrix}\right)$$

and so we see that $Y_1\in N(2,10)$, $Y_2\in N(-1,5)$, and $\text{corr}(Y_1,Y_2)=\frac{1}{\sqrt{2}}$. Thus, by the results in Section 5.6, the distribution of $Y_1|Y_2=y$ is normal with mean $2+\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{10}}{\sqrt{5}}(y-(-1))=y+3$ and variance $10\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)=5$. That is,

$$Y_1|Y_2=y\in N(y+3,5).$$

Problem #13. Let $\mathbf{X}=(X_1,X_2,X_3)'$ where X_1, X_2, X_3 are i.i.d. $N(1,1)$ so that $\mathbf{X}\in N(\boldsymbol{\mu},\boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu}=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad\text{and}\quad\boldsymbol{\Lambda}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Let $\mathbf{Y}=(U,V)'$ where $U=2X_1-X_2+X_3$ and $V=X_1+2X_2+3X_3$. If

$$B=\begin{pmatrix}2&-1&1\\1&2&3\end{pmatrix}$$

then $\mathbf{Y}=B\mathbf{X}$. By Theorem 5.3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu}=\begin{pmatrix}2&-1&1\\1&2&3\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 14 \end{pmatrix}.$$

We can immediately conclude that $U \in N(2,6)$, $V \in N(6,14)$, and $\text{cov}(U,V) = 3$ so that $\text{corr}(U,V) = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}}$. It follows from Section 5.6 that the distribution of $V|U = u$ is

$$N\left(6 + \frac{3}{2\sqrt{21}} \frac{\sqrt{14}}{\sqrt{6}}(u-2), 14\left(1 - \frac{9}{4 \cdot 21}\right)\right).$$

Choosing $u = 3$ therefore implies that

$$V|U = 3 \in N\left(6.5, 12.5\right).$$

Problem #15. Using Theorem 5.3.1, the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$ is

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 & -5 \\ 4 & 9 & -10 \\ -5 & -10 & 13 \end{pmatrix}\right)$$

and so we see that $X_1 \in N(0,2)$, $X_2 \in N(0,9)$, and $X_3 \in N(0,13)$. Since $\text{cov}(X_1, X_3) = -5$, we conclude that $X_1 + X_3 \in N(0,5)$. Finally, we compute $\text{cov}(X_2, X_1 + X_3) = \text{cov}(X_2, X_1) + \text{cov}(X_2, X_3) = 4 - 10 = -6$ and so $\text{corr}(X_2, X_1 + X_3) = -\frac{2}{\sqrt{5}}$. Thus, by the results in Section 5.6, the distribution of $X_2|X_1 + X_3 = x$ is normal with mean $0 - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{5}}(x - 0) = -\frac{6x}{5}$ and variance $9\left(1 - \left(-\frac{2}{\sqrt{5}}\right)^2\right) = \frac{9}{5}$. That is,

$$X_2|X_1 + X_3 = x \in N\left(-\tfrac{6x}{5}, \tfrac{9}{5}\right).$$

Problem #16. Using Theorem 5.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right).$$

By definition,

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{f_{Y_1,Y_2,Y_3}(y,0,0)}{f_{Y_2,Y_3}(0,0)}.$$

From Definition III, we know

$$f_{Y_1,Y_2,Y_3}(y,0,0) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}$$

since

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

The joint distribution of $(Y_2, Y_3)'$ is

$$(Y_2, Y_3)' \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\right)$$

and so

$$f_{Y_2,Y_3}(0,0) = \frac{1}{2\pi\sqrt{3}}.$$

Thus, we conclude

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{\left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}}{\frac{1}{2\pi\sqrt{3}}} = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{2} \exp\left\{-\frac{1}{2}\left(\frac{y}{2/\sqrt{3}}\right)^2\right\}$$

which we recognize as the density function of a normal random variable with mean 0 and variance 4/3. That is,

$$Y_1|Y_2 = Y_3 = 0 \in N\left(0, \tfrac{4}{3}\right).$$

Problem #25. Using Theorem 5.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}9 & 6\\6 & 6\end{pmatrix}\right)$$

and so we see that $Y_1 \in N(3,9)$, $Y_2 \in N(2,6)$, and $\operatorname{corr}(Y_1, Y_2) = \frac{\sqrt{2}}{\sqrt{3}}$. Thus, by the results in Section 5.6, the distribution of $Y_1|Y_2 = 0$ is normal with mean $3 + \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{3}{\sqrt{6}}(0-2) = 1$ and variance $9\left(1 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\right) = 3$. That is,

$$Y_1|Y_2 = 0 \in N(1,3).$$

Problem #39. In order to determine the values of a and b for which $\mathbb{E}(U - a - bV)^2$ is a minimum, we must minimize the function $g(a,b) = \mathbb{E}(U - a - bV)^2$. If $U = X_1 + X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$, then

$$U - a - bV = X_1 + X_2 + X_3 - a - b(X_1 + 2X_2 + 3X_3) = (1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a.$$

Notice that $\mathbb{E}(U - a - bV)^2 = \operatorname{var}(U - a - bV) + [\mathbb{E}(U - a - bV)]^2$. We now compute

$$\begin{aligned}\operatorname{var}(U - a - bV) &= \operatorname{var}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a)\\ &= (1-b)^2 \operatorname{var}(X_1) + (1-2b)^2 \operatorname{var}(X_2) + (1-3b)^2 \operatorname{var}(X_3)\\ &= (1-b)^2 + (1-2b)^2 + (1-3b)^2\end{aligned}$$

using the fact that X_1, X_2, X_3 are i.i.d. $N(1,1)$. Furthermore,

$$\begin{aligned}\mathbb{E}(U - a - bV) = \mathbb{E}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a) &= (1-b) + (1-2b) + (1-3b) - a\\ &= 3 - 6b - a\end{aligned}$$

which implies that

$$g(a,b) = (1-b)^2 + (1-2b)^2 + (1-3b)^2 + [3 - 6b - a]^2 = 12 - 48b + 50b^2 - 6a + 12ab + a^2.$$

To minimize g, we begin by finding the critical points. That is,

$$\frac{\partial}{\partial a} g(a,b) = -6 + 12b + 2a = 0$$

implies $a + 6b = 3$, and

$$\frac{\partial}{\partial b} g(a,b) = -48 + 100b + 12a = 0$$

implies $25b + 3a = 12$. Solving the second equation for b yields

$$25b = 12 - 3a = 12 - 3(3 - 6b) \quad \text{and so} \quad b = \frac{3}{7}.$$

Substituting in gives

$$a = 3 - 6b = 3 - \frac{18}{7} = \frac{3}{7}.$$

Since

$$\frac{\partial^2}{\partial a^2} g(a,b) = 2 > 0$$

and

$$\frac{\partial^2}{\partial a^2} g(a,b) \cdot \frac{\partial^2}{\partial b^2} g(a,b) - \left(\frac{\partial^2}{\partial a \partial b} g(a,b) \right)^2 = 2 \cdot 100 - 12^2 = 56 > 0$$

we conclude by the second derivative test that $a = 3/7$, $b = 3/7$ is indeed the minimum.