Stat 351 Fall 2015
Chapter 2 Solutions

Problem #2. Suppose that $X+Y=2$. By definition of conditional density,

$$f_{X|X+Y=2}(x) = \frac{f_{X,X+Y}(x,2)}{f_{X+Y}(2)}.$$

We now find the joint density $f_{X,X+Y}(x,2)$. Let $U=X$ and $V=X+Y$ so that $X=U$ and $Y=V-U$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Since X and Y are independent $\Gamma(2,a)$, the joint density of (X,Y) is

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} \frac{xy}{a^4}e^{-(x+y)/a}, & \text{for } x>0,\ y>0, \\ 0, & \text{otherwise,} \end{cases}$$

The joint density of (U,V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u,v-u)\cdot|J| = \frac{u(v-u)}{a^4}e^{-v/a}$$

provided that $u>0$ and $v>u$. The marginal density for V is therefore

$$f_V(v) = \int_0^v \frac{u(v-u)}{a^4}e^{-v/a}\,du = a^{-4}e^{-v/a}\int_0^v u(v-u)\,du = \frac{v^3}{6a^4}e^{-v/a}, \quad v>0.$$

Since $V=X+Y$, we can rewrite these densities as $f_{X,X+Y}(x,2) = \frac{x(2-x)}{a^4}e^{-2/a}$, $0<x<2$, and $f_{X+Y}(2) = \frac{2^3}{6a^4}e^{-2/a}$. Finally, we conclude

$$f_{X|X+Y=2}(x) = \frac{f_{X,X+Y}(x,2)}{f_{X+Y}(2)} = \frac{\frac{x(2-x)}{a^4}e^{-2/a}}{\frac{2^3}{6a^4}e^{-2/a}} = \frac{3x(2-x)}{4}$$

provided that $0<x<2$.

Problem #8. (a) The density function for Y is given by

$$f_Y(y) = \int_0^\infty \frac{x^2}{2y^3}\cdot e^{-\frac{x}{y}}\,dx$$

provided that $0<y<1$. Let $u=-\frac{x}{y}$ so that $du=-\frac{1}{y}\,dx$, from which it follows that

$$f_Y(y) = \int_0^\infty \frac{x^2}{2y^3}\cdot e^{-\frac{x}{y}}\,dx = \frac{1}{2}\int_0^\infty u^2e^{-u}\,du = \frac{1}{2}\Gamma(3) = \frac{2!}{2} = 1.$$

That is, $Y\in U(0,1)$.

(b) The conditional density of X given $Y = y$ is therefore

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}}{1} = \frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}$$

provided that $x > 0$. That is, $X|Y = y \in \Gamma(3, y)$.

(c) Since $Y \in U(0,1)$, we know that $E(Y) = \frac{1}{2}$ and $\mathrm{Var}(Y) = \frac{1}{12}$. We also use the fact from page 260 that the mean of a $\Gamma(p, a)$ random variable is pa and the variance is pa^2. Thus, we find that the mean of X is

$$E(X) = E(E(X|Y)) = E(3Y) = 3E(Y) = \frac{3}{2}$$

and the variance of X is

$$\begin{aligned}\mathrm{Var}(X) = \mathrm{Var}(E(X|Y)) + E(\mathrm{Var}(X|Y)) = \mathrm{Var}(3Y) + E(3Y^2) &= 9\,\mathrm{Var}(Y) + 3E(Y^2) \\ &= 9\,\mathrm{Var}(Y) + 3\left[\mathrm{Var}(Y) + (E(Y))^2\right] \\ &= \frac{9}{12} + 3\left(\frac{1}{12} + \frac{1}{4}\right) \\ &= \frac{7}{4}.\end{aligned}$$

Problem #9 (a) Since

$$\int_0^1 \int_0^{1-x} cx\, dy\, dx = c\int_0^1 x(1-x)\, dx = c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{c}{6}$$

we conclude that $c = 6$.

(b) The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{1-y} 6x\, dx = 3(1-y)^2, \quad 0 \le y \le 1,$$

and the marginal for X is

$$f_X(x) = \int_0^{1-x} 6x\, dy = 6x(1-x), \quad 0 \le x \le 1.$$

We conditional densities are then

$$f_{X|Y=y}(x) = \frac{6x}{3(1-y)^2} = \frac{2x}{(1-y)^2}, \quad 0 \le x \le 1-y,$$

and

$$f_{Y|X=x}(y) = \frac{6x}{6x(1-x)} = \frac{1}{1-x}, \quad 0 \le y \le 1-x.$$

Finally, we find

$$E(X|Y=y) = \int_0^{1-y} x \cdot \frac{2x}{(1-y)^2}\, dx = \frac{2}{3}(1-y)$$

and

$$E(Y|X=x) = \int_0^{1-x} y \cdot \frac{1}{1-x}\, dy = \frac{1}{2}(1-x).$$

Problem #10. Since

$$\int_0^1 \int_x^1 cx^2\,dy\,dx = c\int_0^1 x^2(1-x)\,dx = c\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{c}{12}$$

we conclude that $c = 12$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^y 12x^2\,dx = 4y^3, \quad 0 < y < 1,$$

and the marginal for X is

$$f_X(x) = \int_x^1 12x^2\,dy = 12x^2(1-x), \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot 4y^3\,dy = \frac{4}{5}$$

and

$$E(X) = \int_0^1 x \cdot 12x^2(1-x)\,dx = 3 - \frac{12}{5} = \frac{3}{5}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{12x^2}{4y^3} = \frac{3x^2}{y^3}, \quad 0 < x < y,$$

and

$$f_{Y|X=x}(y) = \frac{12x^2}{12x^2(1-x)} = \frac{1}{1-x}, \quad x < y < 1.$$

Finally, we find

$$E(X|Y=y) = \int_0^y x \cdot \frac{3x^2}{y^3}\,dx = \frac{3y}{4}$$

and

$$E(Y|X=x) = \int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}.$$

Problem #11. Since

$$\int_0^1 \int_0^x cx^2y\,dy\,dx = \frac{c}{2}\int_0^1 x^4\,dx = \frac{c}{2}\left[\frac{1}{5}x^5\right]_0^1 = \frac{c}{10}$$

we conclude that $c = 10$. The marginal for Y is therefore given by

$$f_Y(y) = \int_y^1 10x^2y\,dx = \frac{10}{3}y(1-y^3), \quad 0 < y < 1,$$

and the marginal for X is

$$f_X(x) = \int_0^x 10x^2y\,dy = 5x^4, \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot \frac{10}{3}y(1-y^3)\,dy = \frac{10}{9} - \frac{10}{18} = \frac{5}{9}$$

and

$$E(X) = \int_0^1 x \cdot 5x^4 \, dx = \frac{5}{6}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{10x^2 y}{\frac{10}{3}y(1-y^3)} = \frac{3x^2}{1-y^3}, \quad y < x < 1,$$

and

$$f_{Y|X=x}(y) = \frac{10x^2 y}{5x^4} = \frac{2y}{x^2}, \quad 0 < y < x.$$

Finally, we find

$$E(X|Y=y) = \int_y^1 x \cdot \frac{3x^2}{1-y^3} \, dx = \frac{3(1-y^4)}{4(1-y^3)}$$

and

$$E(Y|X=x) = \int_0^x y \cdot \frac{2y}{x^2} \, dy = \frac{2x}{3}.$$

Problem #18. Since

$$\int_0^1 \int_x^1 c(x+y) \, dy \, dx = c \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x^2) \right] dx = c \left[\frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{2}x^3 \right]_0^1 = \frac{c}{2}$$

we conclude that $c = 2$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^y 2(x+y) \, dx = y^2 + 2y^2 = 3y^2, \quad 0 < y < 1,$$

and the marginal for X is

$$f_X(x) = \int_x^1 2(x+y) \, dy = 2x(1-x) + (1-x^2) = 1 + 2x - 3x^2, \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot 3y^2 \, dy = \frac{3}{4}$$

and

$$E(X) = \int_0^1 x \cdot (1 + 2x - 3x^2) \, dx = \frac{1}{2} + \frac{2}{3} - \frac{3}{4} = \frac{5}{12}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{2(x+y)}{3y^2}, \quad 0 < x < y,$$

and

$$f_{Y|X=x}(y) = \frac{2(x+y)}{1+2x-3x^2}, \quad x < y < 1.$$

Finally, we find

$$E(X|Y=y) = \int_0^y x \cdot \frac{2(x+y)}{3y^2} \, dx = \frac{2}{3y^2} \cdot \left(\frac{y^3}{3} + \frac{y^3}{2} \right) = \frac{5y}{9}$$

and

$$E(Y|X=x) = \int_x^1 y \cdot \frac{2(x+y)}{1+2x-3x^2}\,dy = \frac{2}{1+2x-3x^2} \cdot \left(\frac{x(1-x^2)}{2} + \frac{(1-x^3)}{3} \right)$$
$$= \frac{2+3x-5x^3}{3(1+2x-3x^2)}$$
$$= \frac{(2+5x+5x^2)(1-x)}{3(3x+1)(1-x)}$$
$$= \frac{2+5x+5x^2}{3(3x+1)}.$$

Problem #19. Since

$$\int_0^1 \int_{x^2}^x c\,dy\,dx = c \int_0^1 (x-x^2)\,dx = c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{c}{6}$$

we conclude that $c = 6$. The marginal for Y is therefore given by

$$f_Y(y) = \int_y^{\sqrt{y}} 6\,dx = 6(\sqrt{y} - y), \quad 0 \le y \le 1,$$

and the marginal for X is

$$f_X(x) = \int_{x^2}^x 6\,dy = 6(x - x^2) = 6x(1-x), \quad 0 \le x \le 1.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{6}{6(\sqrt{y}-y)} = \frac{1}{\sqrt{y}-y}, \quad y \le x \le \sqrt{y},$$

and

$$f_{Y|X=x}(y) = \frac{6}{6x(1-x)} = \frac{1}{x(1-x)}, \quad x^2 \le y \le x.$$

Finally, we find

$$E(X|Y=y) = \int_y^{\sqrt{y}} x \cdot \frac{1}{\sqrt{y}-y}\,dx = \frac{y-y^2}{2(\sqrt{y}-y)} = \frac{y+\sqrt{y}}{2}$$

and

$$E(Y|X=x) = \int_{x^2}^x y \cdot \frac{1}{x(1-x)}\,dy = \frac{x^2-x^4}{2x(1-x)} = \frac{x(1+x)}{2}.$$

Problem #22. Since

$$\int_0^1 \int_0^{\sqrt{1-x^2}} cx^3y\,dy\,dx = \frac{c}{2}\int_0^1 x^3(1-x^2)\,dx = \frac{c}{2}\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 = \frac{c}{24}$$

we conclude that $c = 24$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{\sqrt{1-y^2}} 24x^3y\,dx = 6y(1-y^2)^2, \quad 0 < y \le 1,$$

and the marginal for X is

$$f_X(x) = \int_0^{\sqrt{1-x^2}} 24x^3y\,dy = 12x^3(1-x^2), \quad 0 < x \leq 1.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{24x^3y}{6y(1-y^2)^2} = \frac{4x^3}{(1-y^2)^2}, \quad 0 < x \leq \sqrt{1-y^2},$$

and

$$f_{Y|X=x}(y) = \frac{24x^3y}{12x^3(1-x^2)} = \frac{2y}{1-x^2}, \quad 0 < y \leq \sqrt{1-x^2}.$$

Finally, we find

$$E(X|Y=y) = \int_0^{\sqrt{1-y^2}} x \cdot \frac{4x^3}{(1-y^2)^2}\,dx = \frac{4(1-y^2)^{5/2}}{5(1-y^2)^2} = \frac{4\sqrt{1-y^2}}{5}$$

and

$$E(Y|X=x) = \int_0^{\sqrt{1-x^2}} y \cdot \frac{2y}{1-x^2}\,dy = \frac{2(1-x^2)^{3/2}}{3(1-x^2)} = \frac{2\sqrt{1-x^2}}{3}.$$

Problem #23. Since

$$\int_0^{1/2}\int_0^{\sqrt{1-4x^2}} cxy\,dy\,dx = \frac{c}{2}\int_0^{1/2} x(1-4x^2)\,dx = \frac{c}{2}\left[\frac{1}{2}x^2 - x^4\right]_0^{1/2} = \frac{c}{32}$$

we conclude that $c = 32$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{\frac{1}{2}\sqrt{1-y^2}} 32xy\,dx = 16y \cdot \frac{1}{4}(1-y^2) = 4y(1-y^2), \quad 0 < y \leq 1,$$

and the marginal for X is

$$f_X(x) = \int_0^{\sqrt{1-4x^2}} 32xy\,dy = 16x(1-4x^2), \quad 0 < x \leq \frac{1}{2}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{32xy}{4y(1-y^2)} = \frac{8x}{1-y^2}, \quad 0 < x \leq \frac{1}{2}\sqrt{1-y^2},$$

and

$$f_{Y|X=x}(y) = \frac{32xy}{16x(1-4x^2)} = \frac{2y}{1-4x^2}, \quad 0 < y \leq \sqrt{1-4x^2}.$$

Finally, we find

$$E(X|Y=y) = \int_0^{\frac{1}{2}\sqrt{1-y^2}} x \cdot \frac{8x}{1-y^2}\,dx = \frac{8 \cdot \frac{1}{8} \cdot (1-y^2)^{3/2}}{3(1-y^2)} = \frac{\sqrt{1-y^2}}{3}$$

and

$$E(Y|X=x) = \int_0^{\sqrt{1-4x^2}} y \cdot \frac{2y}{1-4x^2}\,dy = \frac{2(1-4x^2)^{3/2})}{3(1-4x^2)} = \frac{2\sqrt{1-4x^2}}{3}.$$

Problem #30. If $X|A = a \in W(\frac{1}{a}, \frac{1}{b})$ with $A \in \Gamma(p, \theta)$, then by the law of total probability,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|A=a}(x) f_A(a)\, da = \int_0^{\infty} abx^{b-1}e^{-ax^b} \cdot \frac{1}{\Gamma(p)}\frac{1}{\theta^p}a^{p-1}e^{-a/\theta}\, da$$
$$= \frac{bx^{b-1}}{\theta^p\Gamma(p)} \int_0^{\infty} a^p e^{-ax^b - a/\theta}\, da$$

Let $u = a(x^b + \frac{1}{\theta})$ so that $du = (x^b + \frac{1}{\theta})\, da$ and the integral above becomes

$$= \frac{bx^{b-1}}{\theta^p\Gamma(p)} \int_0^{\infty} u^p (x^b + \frac{1}{\theta})^{-p} e^{-u}\, (x^b + \frac{1}{\theta})^{-1} du = \frac{bx^{b-1}(x^b + \frac{1}{\theta})^{-1-p}}{\theta^p\Gamma(p)} \int_0^{\infty} u^p e^{-u}\, du.$$

But

$$\int_0^{\infty} u^p e^{-u}\, du = \Gamma(p+1)$$

and so we conclude that for $x > 0$ (and using the fact that $\Gamma(p+1) = p \cdot \Gamma(p)$) that

$$f_X(x) = \frac{bx^{b-1}(x^b + \frac{1}{\theta})^{-1-p}}{\theta^p\Gamma(p)}\Gamma(p+1) = \frac{bpx^{b-1}(x^b + \frac{1}{\theta})^{-1-p}}{\theta^p}.$$

The final step is to determine the distribution of X^b. If $Y = X^b$, then

$$P(Y \le y) = P(X^b \le y) = P(X \le y^{1/b})$$

and so

$$f_Y(y) = \frac{1}{b}y^{1/b-1} f_X(y^{1/b}) = \frac{1}{b}y^{1/b-1}\frac{bpy^{(b-1)/b}(y + \frac{1}{\theta})^{-1-p}}{\theta^p} = \frac{p}{\theta^p}\frac{1}{(y + \frac{1}{\theta})^{p+1}}, \quad y > 0,$$

which happens to be the density function of a translated Pareto distribution.