Stat 351 Fall 2015
Assignment #6

Solutions should be completed, but not submitted, by Monday, November 9, 2015.

1. The goal of this problem is to prove the following formula relating the binomial distribution function with the beta distribution function, namely

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = k\binom{n}{k} \int_0^p x^{k-1}(1-x)^{n-k}\,\mathrm{d}x \tag{$*$}$$

where $k \in \{0, 1, \ldots, n\}$ and $0 < p < 1$.

(a) Suppose that

$$I(k,n,p) = k\binom{n}{k} \int_0^p x^{k-1}(1-x)^{n-k}\,\mathrm{d}x.$$

Make the substitution $x = p(1-t)$ to show that

$$I(k,n,p) = k\binom{n}{k} p^k \int_0^1 (1-t)^{k-1}(1-p+pt)^{n-k}\,\mathrm{d}t.$$

(b) Use the binomial formula

$$(a+b)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} a^{m-\ell} b^\ell$$

to show that

$$(1-p+pt)^{n-k} = \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} (1-p)^{n-k-\ell}(pt)^\ell.$$

(c) Substitute the result from **(b)** into the formula from **(a)** to deduce

$$I(k,n,p) = k\binom{n}{k} p^k \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} p^\ell (1-p)^{n-k-\ell} \int_0^1 (1-t)^{k-1} t^\ell\,\mathrm{d}t.$$

(d) Use the fact that

$$\int_0^1 (1-t)^{k-1} t^\ell\,\mathrm{d}t = \frac{\Gamma(k)\Gamma(\ell+1)}{\Gamma(k+\ell+1)}$$

(since it is a beta integral) to conclude that

$$I(k,n,p) = k\binom{n}{k} p^k \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} p^\ell (1-p)^{n-k-\ell} \frac{\Gamma(k)\Gamma(\ell+1)}{\Gamma(k+\ell+1)}.$$

(e) Simplify the factorials in the formula from **(d)** to find

$$I(k,n,p) = \sum_{\ell=0}^{n-k} \frac{n!}{(k+\ell)!(n-(k+\ell))!} p^{k+\ell}(1-p)^{n-(k+\ell)} = \sum_{\ell=0}^{n-k} \binom{n}{k+\ell} p^{k+\ell}(1-p)^{n-(k+\ell)}.$$

(f) Make the change-of-variables $j = k+\ell$ in the index of summation to conclude that **(e)** implies

$$I(k,n,p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

as required.

Remark. Since

$$k\binom{n}{k} = \frac{n!}{(k-1)!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}$$

we see that $(*)$ is equivalent to

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \int_0^p x^{k-1}(1-x)^{n-k}\, \mathrm{d}x.$$

Thus, if $X \in \mathrm{Bin}(n,p)$ and $Y \in \beta(k, n-k+1)$, then this says that

$$P(X \geq k) = P(Y \leq p), \quad \text{or, equivalently,} \quad 1 - F_X(k-1) = F_Y(p).$$

Chapter 4 Homework Hints

- Problem #3, page 113: Consider the cases $y \leq 1/2$ and $y \geq 1/2$ separately.
- Problem #5, page 113:

$$E(F(X_{(n)})) = \int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\, \mathrm{d}y_n.$$

  Use the expression for $f_{X_{(n)}}(y_n)$ derived in class, make the substitution $u = F(y_n)$, and be careful with the limits of integration.

- Problem #16, page 114: Let $U = X_{(1)}$, $V = X_{(2)} - X_{(1)}$, and use techniques of Chapter 1 to find $f_{U,V}(u,v)$ and show that U and V are independent.
- Problem #17, page 114: Consider the cases $x > 1/2$ and $x \leq 1/2$ separately.