Statistics 351 Fall 2009 Quiz #2 – Solutions

1. Let X denote the colour of the ball drawn from the first urn, and let Y denote the colour of the ball drawn from the second urn. Let b denote the unknown number of blue balls in the second urn. We find

$$\begin{aligned}0.44 = P(X = Y) &= P(X = R, Y = R) + P(X = B, Y = B) \\ &= P(X = R)P(Y = R) + P(X = B)P(Y = B) \\ &= \frac{4}{10} \cdot \frac{16}{16+b} + \frac{6}{10} \cdot \frac{b}{16+b}.\end{aligned}$$

That is,

$$6b + 64 = 4.4(16 + b) \quad \text{or, equivalently,} \quad 60b + 64 = 44(16 + b)$$

so that

$$b = \frac{44 \cdot 16 - 64}{16} = 4.$$

The answer is (A). (Problem #4)

2. If N_1, N_2 denote the number of claims received in weeks 1 and 2, respectively, then the probability that exactly seven claims will be received during a given two-week period is given by

$$\begin{aligned}P(N_1 + N_2 = 7) &= \sum_{j=0}^{7} P(N_1 = j, N_2 = 7 - j) = \sum_{j=0}^{7} P(N_2 = 7 - j)P(N_1 = j) \\ &= \sum_{j=0}^{7} \frac{1}{2^{8-j}} \frac{1}{2^{j+1}} = \sum_{j=0}^{7} \frac{1}{2^9} = \frac{8}{2^9} = \frac{1}{64}.\end{aligned}$$

The answer is (D). (Problem #16)

3. If we denote the constant of proportionality by k, then

$$k^{-1} = \int_0^{40} (10 + x)^{-2} dx = -(10 + x)^{-1} \Big|_0^{40} = \frac{1}{10} - \frac{1}{50} = \frac{4}{50} = \frac{2}{25}.$$

Thus, if X denotes the lifetime of the machine part, then

$$P(X \leq 6) = \int_0^6 \frac{25}{2}(10 + x)^{-2} dx = \frac{-25(10 + x)^{-1}}{2} \Big|_0^6 = \frac{25}{20} - \frac{25}{32} = \frac{15}{32} = 0.46875.$$

The answer is (C). (Problem #34)

4. The conditional probability that V exceeds 40,000, given that V exceeds 10,000, is given by

$$P(V \geq 40000 \,|\, V \geq 10000) = P(Y \geq 0.4 \,|\, Y \geq 0.1) = \frac{P(Y \geq 0.4, Y \geq 0.1)}{P(Y \geq 0.1)} = \frac{P(Y \geq 0.4)}{P(Y \geq 0.1)}$$

$$= \frac{\int_{0.4}^{1} k(1-y)^4 dy}{\int_{0.1}^{1} k(1-y)^4 dy} = \frac{\int_{0.4}^{1} (1-y)^4 dy}{\int_{0.1}^{1} (1-y)^4 dy} = \frac{-\frac{1}{5}(1-y)^5 \Big|_{0.4}^{1}}{-\frac{1}{5}(1-y)^5 \Big|_{0.1}^{1}}$$

$$= \frac{0^5 - (0.6)^5}{0^5 - (0.9)^5} = \frac{2^5}{3^5} = \frac{32}{243} \doteq 0.1316872.$$

The answer is (B). (Problem #36)

5. Let T denote time of failure (measured in years) so that the density function for T is

$$f(t) = \frac{1}{10} e^{-t/10}, \quad t > 0.$$

If Y denotes the payment of the company, then

$$\begin{aligned} E(Y) &= xP(Y = x) + 0.5xP(Y = 0.5x) = xP(T \leq 1) + 0.5xP(1 \leq T \leq 3) \\ &= x \int_0^1 \frac{1}{10} e^{-t/10} dt + 0.5x \int_0^1 \frac{1}{10} e^{-t/10} dt \\ &= -xe^{-t/10} \Big|_0^1 - 0.5xe^{-t/10} \Big|_1^3 \\ &= x(1 - e^{-1/10}) + 0.5x(e^{-1/10} - e^{-3/10}) \\ &= x(1 - 0.5e^{-1/10} - 0.5e^{-3/10}) \end{aligned}$$

If the expected payment the insurance company makes is $E(Y) = 1000$, then x satisfies

$$x = \frac{1000}{(1 - 0.5e^{-1/10} - 0.5e^{-3/10})} \doteq 5644.227.$$

The answer is (D). (Problem #47)

6. Notice that $(x^2 - 2x + 2)/2$ evaluated at $x = 1$ is equal to $1/2$. Thus, the distribution function for X has a jump at $x = 1$ implying that $P(X = 1) = 1/2$. If we differentiate $F(x)$, then $f(x) = 2x - 2$ for $1 \leq x \leq 2$ (noting that $P(1 \leq X \leq 2) = 1/2$) so that

$$E(X) = 1P(X = 1) + \frac{1}{2} \int_1^2 x(2x-2)dx = \frac{1}{2} + \frac{1}{2}\left[\frac{2}{3}x^3 - x^2\right]_1^2 = \frac{1}{2} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{4}{3}$$

and

$$E(X^2) = 1^2 P(X = 1) + \frac{1}{2} \int_1^2 x^2(2x-2)dx = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}x^4 - \frac{2}{3}x^3\right]_1^2 = \frac{1}{2} + 4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \frac{23}{12}$$

3/12 +48/12 -28/12 Thus, the variance of X is

$$\mathrm{var}(X) = E(X^2) - [E(X)]^2 = \frac{23}{12} - \frac{16}{9} = \frac{5}{36}.$$

The answer is (C). (Problem #62.)