Statistics 351 Fall 2009 Midterm #2 – Solutions

1. Since X_1, X_2, and X_3 are independent and normally distributed, we conclude that if we set $\mathbf{X} = (X_1, X_2, X_3)'$, then $\mathbf{X}$ is multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X} + \mathbf{b}$. By Theorem 5.3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} + \mathbf{b} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}.$$

2. By Theorem 4.2.1, the joint density of the maximum and minimum, we find

$$f_{X_{(1)},X_{(4)}}(y_1, y_4) = 12(y_4 - y_1)^2$$

provided that $0 < y_1 < y_4 < 1$. Therefore,

$$P(X_{(1)} + X_{(4)} \le 1) = \iint\limits_{\left\{\substack{0<x+y<1, \\ 0<x<y<1}\right\}} f_{X_{(1)},X_{(4)}}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Drawing the region of integration $\{0 < x + y < 1, 0 < x < y < 1\}$, we see that it can be described by $0 < x < 1/2$ and $x < y < 1 - x$. This gives

$$\begin{aligned} P(X_{(1)} + X_{(4)} \le 1) &= \int_0^{1/2} \int_x^{1-x} f_{X_{(1)},X_{(4)}}(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{1/2} \int_x^{1-x} 12(y - x)^2\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{1/2} 4(y - x)^3 \Big|_{y=x}^{y=1-x} \mathrm{d}x = \int_0^{1/2} 4(1 - 2x)^3\,\mathrm{d}x = -\frac{1}{2}(1 - 2x)^4 \Big|_0^{1/2} = \frac{1}{2}. \end{aligned}$$

3. Recall that if $X \in U(0, 1)$, then $E(X) = 1/2$ and $\operatorname{var}(X) = 1/12$. Since the conditional distribution of $Y|X = x \in N(x, x^2)$, we know that $E(Y|X) = X$ and $\operatorname{var}(Y|X) = X^2$. Therefore, it follows from Theorem 2.2.1 that

$$E(Y) = E(E(Y|X)) = E(X) = \frac{1}{2}.$$

From Corollary 2.2.3.1, we find

$$\mathrm{var}(Y) = E(\mathrm{var}(Y|X)) + \mathrm{var}(E(Y|X)) = E(X^2) + \mathrm{var}(X) = \frac{1}{12} + \left(\frac{1}{2}\right)^2 + \frac{1}{12} = \frac{5}{12}.$$

By definition, $\mathrm{cov}(X, Y) = E(XY) + E(Y)E(X)$. In order to calculate $E(XY)$, notice that Theorem 2.2.1 implies $E(XY) = E(E(XY|X))$. However, by Theorem 2.2.2 ("taking out what is known"), we see that

$$E(E(XY|X)) = E(XE(Y|X)) = E(X \cdot X) = E(X^2) = \frac{1}{12} + \left(\frac{1}{2}\right)^2.$$

Combining everything gives

$$\mathrm{cov}(X, Y) = E(XY) - E(Y)E(X) = \frac{1}{12} + \left(\frac{1}{2}\right)^2 - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

4. Recall that a covariance matrix must be symmetric and non-negative definite. Since the diagonal entries represent variances, they must be non-negative. Hence, we immediately see that A and D cannot be covariance matrices since A has a negative diagonal entry and D is not symmetric. If we check the upper left blocks of B, we find $\det[B_1] = 4$, $\det[B_2] = 8$, and $\det[B_3] = 4$. Thus, B is positive definite (as well as symmetric and having positive diagonal entries) and so it can be a covariance matrix. If we check the upper left blocks of C, we find $\det[C_1] = 1$ and $\det[C_2] = -3$ so that C cannot be a covariance matrix.

5. Since $\mathbf{X}$ has a multivariate normal distribution, we know from Definition I that X_1+X_2 is normal with

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 0 + 0 = 0$$

and

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_1, X_2) = 1 + 2 + 2(-1) = 1.$$

Since $\mathbf{Y}$ has a multivariate normal distribution, we know from Definition I that Y_1+Y_2 is normal with

$$E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 1 + 1 = 2$$

and

$$\mathrm{var}(Y_1 + Y_2) = \mathrm{var}(Y_1) + \mathrm{var}(Y_2) + 2\,\mathrm{cov}(Y_1, Y_2) = 2 + 3 + 2(-2) = 1.$$

Furthermore, since $\mathbf{X}$ and $\mathbf{Y}$ are independent, we conclude that $X_1 + X_2$ and $Y_1 + Y_2$ are independent. That is, $Z = (X_1+X_2)-(Y_1+Y_2)$ is the sum of independent normal random variables and so it must also be normal. Finally, we calculate

$$E(Z) = E((X_1 + X_2) - (Y_1 + Y_2)) = E(X_1 + X_2) - E(Y_1 + Y_2) = 0 - 2 = -2$$

and

$$\mathrm{var}(Z) = \mathrm{var}((X_1 + X_2) - (Y_1 + Y_2)) = \mathrm{var}(X_1 + X_2) + \mathrm{var}(Y_1 + Y_2) = 1 + 1 = 2.$$

That is, $Z \in N(-2, 2)$.