Statistics 351 (Fall 2009)
Independence of $\overline{X}$ and S^2 in a Normal Sample

The goal of this lecture is to prove that $\overline{X}$ and S^2 are independent for a normal sample; our proof of this theorem will follow Example 5.8.3.

Theorem. *Suppose that $X_1, \ldots, X_n$ are independent $\mathcal{N}(0,1)$ random variables. If*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \textit{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

denote the sample mean *and* sample variance*, respectively, then $\overline{X}$ and S^2 are independent.*

Proof. Since $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(0,1)$, we conclude (using, say moment generating functions) that $\overline{X} \in \mathcal{N}(0, 1/n)$. Similarly, we can show that

$$\begin{aligned}\overline{X} - X_j = \frac{1}{n}(X_1 + \cdots + X_{j-1} + X_{j+1} + \cdots + X_n) - \frac{n-1}{n}X_j &\in \mathcal{N}\left(0, \frac{n-1}{n^2} + \frac{(n-1)^2}{n^2}\right)\\ &= \mathcal{N}\left(0, \frac{n-1}{n}\right)\end{aligned}$$

and so

$$X_j - \overline{X} \in \mathcal{N}\left(0, \frac{n-1}{n}\right)$$

as well. We also note that

$$\mathrm{Cov}(X_j, \overline{X}) = \mathrm{Cov}\left(X_j, \frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}(X_j, X_i) = \frac{1}{n}\,\mathrm{Cov}(X_j, X_j) = \frac{1}{n},$$

and so for $j \neq k$ it follows that

$$\begin{aligned}\mathrm{Cov}(X_j - \overline{X}, X_k - \overline{X}) &= \mathrm{Cov}(X_j, X_k) - \mathrm{Cov}(X_j, \overline{X}) - \mathrm{Cov}(\overline{X}, X_k) + \mathrm{Cov}(\overline{X}, \overline{X})\\ &= 0 - \frac{1}{n} - \frac{1}{n} + \frac{1}{n}\\ &= -\frac{1}{n}\end{aligned}$$

using the fact that $\mathrm{Cov}(\overline{X}, \overline{X}) = \mathrm{Var}(\overline{X}) = 1/n$. Similarly,

$$\begin{aligned}\mathrm{Cov}(\overline{X}, X_j - \overline{X}) &= \mathrm{Cov}(X_j, \overline{X}) - \mathrm{Cov}(\overline{X}, \overline{X})\\ &= \frac{1}{n} - \frac{n}{n^2}\\ &= 0.\end{aligned}$$

Thus, we see that $(\overline{X}, X_1 - \overline{X}, \ldots, X_n - \overline{X})' \in \mathcal{N}(\overline{0}, \mathbf{\Lambda})$ where

$$\mathbf{\Lambda} = \begin{bmatrix} 1/n & 0 & 0 & \cdots & 0\\ 0 & (n-1)/n & -1/n & \cdots & -1/n\\ 0 & -1/n & (n-1)/n & \cdots & -1/n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -1/n & -1/n & \cdots & (n-1)/n \end{bmatrix}.$$

By Theorem 5.7.2, we conclude from the form of $\mathbf{\Lambda}$ that $\overline{X}$ and $(X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent normal random vectors. It now follows from the transformation theorem (Theorem 1.2.1) that since $\overline{X}$ and $\mathbf{X} = (X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent, so too are $\overline{X}$ and $\mathbf{X}'\mathbf{X}$. Since

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} (X_i - \overline{X})^2,$$

this implies that $\overline{X}$ and S^2 are independent. □

Suppose that $Y_1, \ldots, Y_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$. We can use the previous result to show that

$$\overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{and} \quad S_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

are also independent. If we define $X_i = (Y_i - \mu)/\sigma$, then $X_i \in \mathcal{N}(0,1)$. Therefore,

$$\overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right) = \frac{\overline{X} - \mu}{\sigma}$$

and

$$S_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} - \frac{\overline{X} - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{S^2}{\sigma^2}.$$

Thus, since $\overline{X}$ and S^2 are independent, so too are

$$\frac{\overline{X} - \mu}{\sigma} = \overline{Y} \quad \text{and} \quad \frac{S^2}{\sigma^2} = S_Y^2.$$