Make sure that this examination has 13 numbered pages

University of Regina
Department of Mathematics & Statistics
Final Examination
200830
(December 12, 2008)

Statistics 351
Intermediate Probability

Name: ______________________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

Read all of the following information before starting the exam.

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. **Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.***

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*This test has **13** numbered pages with **11** questions totalling **150** points. The number of points per question is indicated. For questions with multiple parts, all parts are equally weighted.*

Fact: For $\lambda > 0$, the density of a random variable $X \in \text{Exp}(\lambda)$ is

$$f_X(x) = \frac{1}{\lambda} \exp\left\{-\frac{x}{\lambda}\right\}, \quad 0 < x < \infty.$$

Fact: For $p > 0$, the Gamma function is given by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx.$$

Fact: You may find it useful to know that for any $a > 0$ integration-by-parts implies

$$\int a^2 x e^{-ax} \, dx = -axe^{-ax} - e^{-ax}.$$

DO NOT WRITE BELOW THIS LINE

Problem 1	________	Problem 2	________	Problem 3	________
Problem 4	________	Problem 5	________	Problem 6	________
Problem 7	________	Problem 8	________	Problem 9	________
Problem 10	________	Problem 11	________		
				TOTAL	________

1. (*28 points*) Suppose that a random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & \text{if } 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that $f_{X,Y}$ is, in fact, a density.

(b) Find $f_X(x)$, the marginal density function of X.

(c) Use your result of **(b)** to compute $E(X)$.

(d) Find $f_{Y|X=x}(y)$, the conditional density function of $Y|X = x$.

(continued)

(e) Compute $E(Y|X = x)$.

(f) Use your results of **(b)** and **(e)** to compute $E(Y)$.

(g) Let $U = XY$. Determine $f_U(u)$, the density function of U

2. (*12 points*) Suppose that the random variable $Y \in \Gamma(p, 1/a)$ with $p > 0$ and $a > 0$ so that the density function of Y is

$$f_Y(y) = \frac{a^p}{\Gamma(p)} y^{p-1} e^{-ay}, \quad y > 0.$$

Suppose further that the conditional distribution of X given $Y = y$ is Weibull with parameter $\theta > 0$ so that the density function of $X|Y = y$ is

$$f_{X|Y=y}(x) = \theta y x^{\theta-1} \exp\{-yx^\theta\}, \quad x > 0.$$

(a) Determine $f_X(x)$, the density function of X.

(b) Suppose that $Z = X^\theta$. Determine $f_Z(z)$, the density function of Z.

3. (*8 points*) Suppose that the random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x, y) = \frac{x}{27} \exp\left\{-\frac{x+y}{3}\right\}, \quad x > 0,\ y > 0.$$

Compute $P\{X < Y\}$.

4. (*12 points*) A point X is chosen uniformly from the interval $(0, 1)$. A second point Y is chosen uniformly from the interval $(X, 1)$.

(a) Determine the joint density function of $(X, Y)'$.

(b) Compute $\text{Var}(Y)$.

5. (*16 points*) Suppose that X_1 and X_2 are independent and identically distributed $\Gamma(2, 1)$ random variables so that

$$f_{X_1}(x) = f_{X_2}(x) = xe^{-x}, \quad x > 0.$$

As usual, let $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$, denote the minimum and maximum, respectively, of X_1 and X_2.

(a) Determine $f_{X_{(1)}}(y)$, the density function of $X_{(1)}$.

(b) Determine $f_{X_{(2)}}(y)$, the density function of $X_{(2)}$.

(continued)

(c) Determine $f_{X_{(1)},X_{(2)}}(y_1, y_2)$, the joint density function of $(X_{(1)}, X_{(2)})'$.

(d) Define the random variable U by

$$U = \frac{X_{(1)}}{X_{(2)}} = \frac{\min\{X_1, X_2\}}{\max\{X_1, X_2\}}$$

to be the ratio of the minimum to the maximum. Determine $f_U(u)$, the density function of U.

6. (*28 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \frac{3}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{5}{2} \end{bmatrix}.$$

(a) Determine the eigenvalues of $\boldsymbol{\Lambda}$.

(b) Find an orthogonal matrix C and a diagonal matrix D such that $CDC' = \boldsymbol{\Lambda}$.

(c) Determine $f_{\mathbf{X}}(x_1, x_2)$, the density function of $\mathbf{X}$.

(d) Determine the distribution of $X_2 | X_1 = 0$.

(continued)

(e) Define the random vector $\mathbf{Y} = (Y_1, Y_2)'$ by setting $\mathbf{Y} = C'\mathbf{X}$ where C is the orthogonal matrix you found in **(b)**. Determine the distribution of $\mathbf{Y}$.

(f) Explain why the random variables Y_1 and Y_2 are independent.

(g) Determine the standard form of the level curves of the density function $f_{\mathbf{X}}(x_1, x_2)$. Normalize your quadratic form so that it is in standard form, i.e., so that $Q(\mathbf{x}) = Q(\mathbf{y}) = 1$. Express your answer analytically, geometrically, and descriptively.

7. (*8 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has a multivariate normal distribution $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where $\rho = \text{Corr}(X_1, X_2)$ satisfies $0 < |\rho| < 1$. Define the random vector $\mathbf{Y} = (Y_1, Y_2)'$ by setting

$$Y_1 = \frac{X_1}{\sigma_1} - \rho\frac{X_2}{\sigma_2} \quad \text{and} \quad Y_2 = \frac{X_2}{\sigma_2}.$$

(a) Determine the distribution of $\mathbf{Y}$.

(b) Explain why Y_1 and Y_2 are independent.

8. (*5 points*) Suppose that $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find a matrix A such that $\mathbf{X}'A\mathbf{X}$ has a $\chi^2(3)$ distribution.

9. (*8 points*) Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = \frac{1}{2}$. Set $S_0 = 0$ and for $n = 1, 2, 3, \ldots$, define the random variable S_n by

$$S_n = \sum_{j=1}^{n} Y_j$$

so that $\{S_n, n = 0, 1, 2, \ldots\}$ a simple random walk starting at 0. Furthermore, as shown in class, $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale. Define the process $\{M_n, n = 0, 1, 2, \ldots\}$ by setting

$$M_n = S_n^3 - 3nS_n.$$

Show that $\{M_n, n = 0, 1, 2, \ldots\}$ is a martingale.

Note: It is equivalent to show $E(M_{n+1}|M_n) = M_n$ *or* $E(M_{n+1}|S_n) = M_n$ *or* $E(M_{n+1}|Y_n) = M_n$.

10. (*10 points*) The purpose of this problem is to derive another version of the gambler's ruin result studied in class. As in the previous problem, suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = \frac{1}{2}$. Set $S_0 = 0$ and for $n = 1, 2, 3, \ldots$, define the random variable S_n by

$$S_n = \sum_{j=1}^{n} Y_j$$

so that $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk starting at 0. If we define $X_n = S_n^2 - n$, then as shown in class, both $\{S_n, n = 0, 1, 2, \ldots\}$ and $\{X_n, n = 0, 1, 2, \ldots\}$ are martingales.

Let $a > 0$, $b > 0$ be positive integers and let T be the first time that the simple random walk reaches either $-a$ or b; formally,

$$T = \min\{n \geq 0 : S_n = -a \text{ or } S_n = b\}.$$

The random time T is a stopping time, and you may assume that the optional sampling theorem may be applied to both the martingales $\{S_n, n = 0, 1, 2, \ldots\}$ and $\{X_n, n = 0, 1, 2, \ldots\}$.

(a) Carefully verify that $P(S_T = b) = \dfrac{a}{a+b}$.

(b) Carefully verify that $E(T) = ab$.

11. (*15 points*) In New York City, subway trains are notoriously unreliable. In fact, subway trains arrive at Grand Central Station according to a Poisson process with a rate (or intensity) of 1 train every 4 minutes.

(a) How many subway trains are expected to arrive in one hour? (Recall that there are 60 minutes in one hour.)

(b) Suppose that Christian is going to work and arrives at Grand Central Station at 8:00 a.m. just as a subway train is departing. What is the probability that Christian will wait at least 8 minutes for the next train to arrive?

(c) Suppose that, after work, Christian and Veronica have agreed to meet at Grand Central Station at 5:00 p.m. Christian is punctual and arrives at 5:00 p.m. However, Veronica is late leaving work and so she arrives at Grand Central Station at 5:16 p.m. What is the probability that at least 3 subway trains pass Christian while he is waiting for Veronica?

The End. Happy Holidays!