Stat 351 Fall 2009
Assignment #1 Solutions

Problem 1:

(a) If $X \sim \text{Unif}[1,3]$, then $F_X(x) = \frac{x-1}{2}$ for $1 \le x \le 3$, and if $Y \sim \mathcal{N}(0,1)$, then

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

for $-\infty < y < \infty$. Since X and Y are independent, we conclude that

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) = \frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

for $1 \le x \le 3$ and $-\infty < y < \infty$. We should also note that if $x < 1$, then $F_X(x) = 0$ and if $x \ge 3$, then $F_X(x) = 1$. Combining everything we conclude

$$F_{X,Y}(x,y) = \begin{cases} \frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, & \text{if } 0 \le x \le 2 \text{ and } -\infty < y < \infty, \\ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, & \text{if } x > 3 \text{ and } -\infty < y < \infty, \\ 0, & \text{if } x < 1 \text{ and } -\infty < y < \infty. \end{cases}$$

(b) We find

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} \left[\frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, \right] = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Since $f_X(x) = \frac{1}{2}$, $1 \le x \le 3$, and $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $-\infty < y < \infty$, we see that

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

as required.

(c) If $Z \in \text{Exp}(4)$ is independent of X and Y, then the joint density of $(X,Y,Z)'$ is given by

$$f_{X,Y,Z}(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot \frac{1}{4} e^{-z/4} = \frac{1}{\sqrt{128\pi}}\, e^{-\frac{1}{4}(z+2y^2)}$$

for $1 \le x \le 3$, $-\infty < y < \infty$, and $z > 0$.

Problem 2:

(a) Observe that $\text{cov}(X,Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) = \mathbb{E}(XZ)$ since $\mathbb{E}(X) = 0$. But $\mathbb{E}(XZ) = \mathbb{E}(X \cdot YX) = \mathbb{E}(X^2 Y) = \mathbb{E}(X^2)\mathbb{E}(Y) = 0$ using the assumed independence of Y and X. Hence, we conclude that $\text{cov}(X,Z) = 0$.

(b) We see that

$$\begin{aligned} P\{Z \ge 1\} = P\{XY \ge 1\} &= P\{X \ge 1, Y = 1\} + P\{X \le -1, Y = -1\} \\ &= P\{X \ge 1\}P\{Y = 1\} + P\{X \le -1\}P\{Y = -1\} \\ &= \frac{1}{2}P\{X \ge 1\} + \frac{1}{2}P\{X \le -1\} \\ &= P\{X \ge 1\} \end{aligned}$$

using the symmetry of the normal distribution.

Since

$$P\{X \geq 1, Z \geq 1\} = P\{X \geq 1, XY \geq 1\} = P\{X \geq 1, Y = 1\} = \frac{1}{2}P\{X \geq 1\}$$

and since

$$P\{Z \geq 1\} \in (0, 1/2),$$

we conclude that

$$P\{X \geq 1, Z \geq 1\} \neq P\{X \geq 1\}P\{Z \geq 1\}$$

which implies that X and Z are not independent. (Note that $P\{X \geq 1\} = P\{Z \geq 1\} \doteq 0.1587$.)

(c) As in **(b)** we have

$$\begin{aligned}
P\{Z \geq x\} = P\{XY \geq x\} &= P\{X \geq x, Y = 1\} + P\{X \leq -x, Y = -1\} \\
&= P\{X \geq x\}P\{Y = 1\} + P\{X \leq -x\}P\{Y = -1\} \\
&= \frac{1}{2}P\{X \geq x\} + \frac{1}{2}P\{X \leq -x\} \\
&= P\{X \geq x\}
\end{aligned}$$

using the symmetry of the normal distribution. Since $P\{X \geq x\} = P\{Z \geq x\}$ is equivalent to saying $P\{X \leq x\} = P\{Z \leq x\}$ which in turn is equivalent to saying that $F_X(x) = F_Z(x)$, we conclude that X and Z have the same distribution (i.e., $Z \in \mathcal{N}(0,1)$).

Problem 3 (Exercise 1.2): This exercise was discussed in class; we just complete the missing details. Since $f_{X,Y}(x,y) = 1/\pi$ for $x^2 + y^2 \leq 1$, we have

$$\mathbb{E}(XY) = \iint\limits_{\{x^2+y^2\leq 1\}} xy \cdot \frac{1}{\pi} \cdot \, dx\, dy.$$

To compute this double integral, we use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq r \leq 1$, $0 \leq \theta < 2\pi$, $dx\, dy = r\, dr\, d\theta$, and so

$$\begin{aligned}
\mathbb{E}(XY) = \iint\limits_{\{x^2+y^2\leq 1\}} xy \cdot \frac{1}{\pi} \cdot \, dx\, dy &= \int_0^{2\pi}\int_0^1 r\cos\theta \cdot r\sin\theta \cdot \frac{1}{\pi} \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{r^3}{\pi}\cos\theta\sin\theta\, dr\, d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos\theta\sin\theta\, d\theta \\
&= \frac{1}{8\pi}\int_0^{2\pi} \sin(2\theta)\, d\theta \\
&= \frac{1}{16\pi}\cos(2\theta)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Furthermore, we find

$$\mathbb{E}(X) = \int_{-1}^{1} x \cdot \frac{2}{\pi}\sqrt{1-x^2}\, dx \quad \text{and} \quad \mathbb{E}(Y) = \int_{-1}^{1} y \cdot \frac{2}{\pi}\sqrt{1-y^2}\, dy.$$

Therefore, since both of these integrals are the same, we only need to evaluate one of them. Thus, letting $u = 1 - x^2$ so that $du = -2x\,dx$, we find

$$\mathbb{E}(Y) = \mathbb{E}(X) = \int_{-1}^{1} x \cdot \frac{2}{\pi}\sqrt{1-x^2}\,dx = -\frac{1}{\pi}\int_{0}^{0}\sqrt{u}\,du = 0.$$

Hence, we conclude that $\operatorname{cov}(X,Y) = \mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y) = 0$ and so X and Y are, in fact, dependent but uncorrelated random variables.

Problem 4 (Exercise 1.3): If $(X,Y)'$ is uniformly distributed on the square with corners $(\pm 1, \pm 1)$, then the joint density of $(X,Y)'$ is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & \text{if } -1 \le x \le 1, -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

- The marginal density of X is given by

  $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

  If $-1 \le x \le 1$, then the range of possible y values is $-1 \le y \le 1$, and so

  $$f_X(x) = \int_{-1}^{1} \frac{1}{4}\,dy = \frac{1}{2}.$$

  That is,

  $$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

  Similarly, if $-1 \le y \le 1$, then the range of possible x values is $-1 \le x \le 1$, and so

  $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{2}.$$

  That is,

  $$f_Y(y) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

  Since $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$, we conclude that X and Y are independent.

- If X and Y are independent, then they are necessarily uncorrelated since $E(XY) = E(X)E(Y)$ so that

  $$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$$

Problem 5 (Exercise 1.1): Since the volume of the unit sphere in $\mathbb{R}^3$ is $4\pi/3$, the joint density of $(X,Y,Z)'$ is

$$f_{X,Y,Z}(x,y,z) = \begin{cases} \frac{3}{4\pi}, & \text{if } x^2+y^2+z^2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

- Therefore, the marginal density of $(X,Y)'$ is given by

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz.$$

If x, y, z are constrained to have $x^2+y^2+z^2 \le 1$, then for fixed x with $-1 \le x \le 1$, the range of possible y values is $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, and that the range of z is $-\sqrt{1-x-y^2} \le z \le \sqrt{1-x^2-y^2}$. It therefore follows that

$$f_{X,Y}(x,y) = \int_{-\sqrt{1-x-y^2}}^{\sqrt{1-x^2-y^2}} \frac{3}{4\pi}\,dz = \frac{3}{2\pi}\sqrt{1-x^2-y^2}$$

for $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. In other words,

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2\pi}\sqrt{1-x^2-y^2}, & \text{if } x^2+y^2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

- The marginal density of X is then given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz\,dy.$$

From our work above, we find that if $-1 \le x \le 1$, then

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy.$$

This can be solved with a u-substitution. Let $y = \left(\sqrt{1-x^2}\right)\cdot \sin u$ so that

$$dy = \left(\sqrt{1-x^2}\right)\cdot \cos u\,du$$

and so

$$\begin{aligned} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy &= \frac{3}{2\pi}(1-x^2)\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \left(\sqrt{1-\sin^2 u}\right)\cdot \cos u\,du \\ &= \frac{3}{2\pi}(1-x^2)\int_{-\pi/2}^{\pi/2} \cos^2 u\,du. \end{aligned}$$

being careful to watch our new limits of integration and remembering that $\sin^{-1}(-1) = -\pi/2$ and $\sin^{-1}(1) = \pi/2$. Recalling the half-angle identities for cosine, we find

$$\int \cos^2 u\,du = \int \frac{1}{2} + \frac{1}{2}\cos(2u)\,du = \frac{u}{2} + \frac{1}{4}\sin(2u)$$

and so

$$\begin{aligned} \frac{3}{2\pi}(1-x^2)\int_{-\pi/2}^{\pi/2} \cos^2 u\,du &= \frac{3}{2\pi}(1-x^2)\left[\frac{u}{2} + \frac{1}{4}\sin(2u)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{3}{2\pi}(1-x^2)\left[\frac{\pi/2}{2} - \frac{-\pi/2}{2}\right] \\ &= \frac{3}{4}(1-x^2). \end{aligned}$$

In summary,

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2), & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note: You can check that f_X is, in fact, a density by verifying that

$$\int_{-1}^{1} \frac{3}{4}(1-x^2)\, dx = 1.$$

Problem 6: Since X_1, X_2, X_3 are independent and identically distributed, by can immediately conclude by symmetry that the 6 events

$$\{X_1 < X_2 < X_3\},\ \{X_1 < X_3 < X_2\},\ \{X_2 < X_1 < X_3\},$$

$$\{X_2 < X_3 < X_1\},\ \{X_3 < X_1 < X_2\},\ \{X_3 < X_2 < X_1\}$$

are equally likely. Since X_1, X_2, X_3 are continuous random variables, we know that events such as $\{X_1 = X_2\}$ have probability zero. Thus, we conclude that these six events are exhaustive; that is,

$$\begin{aligned} P\{X_1 < X_2 < X_3\} &= P\{X_1 < X_3 < X_2\} = P\{X_2 < X_1 < X_3\} \\ &= P\{X_2 < X_3 < X_1\} = P\{X_3 < X_1 < X_2\} = P\{X_3 < X_2 < X_1\} \\ &= \frac{1}{6}. \end{aligned}$$

It now follows that

(a) $P\{X_1 > X_2\} = P\{X_2 < X_1 < X_3\} + P\{X_2 < X_3 < X_1\} + P\{X_3 < X_2 < X_1\} = \dfrac{1}{2}$,

(b) $P\{X_1 > X_2 | X_1 > X_3\} = P\{X_2 < X_3 < X_1\} + P\{X_3 < X_2 < X_1\} = \dfrac{2}{3}$,

(c) $P\{X_1 > X_2 | X_1 < X_3\} = P\{X_2 < X_1 < X_3\} = \dfrac{1}{6}$.