

Problem #2. Let $\mathbf{X} = (X, Y)'$ with

$$\mathbf{X} \in N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right),$$

and consider the change of variables to polar coordinates $(R, \Theta)'$. The inverse of this transformation is given by

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

for $0 \leq \theta < 2\pi$, $r > 0$ so that the Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

Since the density of $(X, Y)'$ is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\}, \quad -\infty < x, y < \infty,$$

it now follows from Theorem 1.2.1 that the density of $(R, \Theta)'$ is

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r \cos \theta, r \sin \theta) \cdot |J| \\ &= r f_{X,Y}(r \cos \theta, r \sin \theta) \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)}(r^2 \cos^2 \theta - 2\rho r^2 \sin \theta \cos \theta + r^2 \sin^2 \theta) \right\} \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{r^2(1-\rho \sin 2\theta)}{2(1-\rho^2)} \right\} \end{aligned}$$

for $0 \leq \theta < 2\pi$, $r > 0$. The marginal density for Θ is therefore given by

$$\begin{aligned} f_{\Theta}(\theta) &= \int_0^{\infty} \frac{r}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{r^2(1-\rho \sin 2\theta)}{2(1-\rho^2)} \right\} dr \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\infty} r \exp \left\{ -\frac{r^2(1-\rho \sin 2\theta)}{2(1-\rho^2)} \right\} dr. \end{aligned}$$

Making the change of variables

$$u = \frac{r^2(1-\rho \sin 2\theta)}{2(1-\rho^2)} \quad \text{so that} \quad \frac{(1-\rho^2) du}{(1-\rho \sin 2\theta)} = r dr$$

implies that

$$f_{\Theta}(\theta) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \frac{(1-\rho^2)}{(1-\rho \sin 2\theta)} \int_0^{\infty} e^{-u} du = \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho \sin 2\theta)}$$

provided $0 \leq \theta < 2\pi$.

Problem #4. If the random vector $(X, Y)'$ has a multivariate normal distribution, then it follows from Definition I that both $X + Y$ and $X - Y$ are normal random variables. If $\text{var}(X) = \text{var}(Y)$, then

$$\text{cov}(X + Y, X - Y) = \text{cov}(X, X) - \text{cov}(X, Y) + \text{cov}(Y, X) + \text{cov}(Y, Y) = \text{var}(X) - \text{var}(Y) = 0.$$

Theorem 5.7.1 therefore implies that $X + Y$ and $X - Y$ are independent since $\text{cov}(X + Y, X - Y) = 0$.

Problem #11. Note that by Theorem 5.7.1, in order to show X_1 , X_2 , and X_3 are independent, it is enough to show that $\text{cov}(X_1, X_2) = \text{cov}(X_1, X_3) = \text{cov}(X_2, X_3) = 0$. Thus, if X_1 and $X_2 + X_3$ are independent, then $\text{cov}(X_1, X_2 + X_3) = \text{cov}(X_1, X_2) + \text{cov}(X_1, X_3) = 0$ and so

$$\text{cov}(X_1, X_2) = -\text{cov}(X_1, X_3). \quad (1)$$

If X_2 and $X_1 + X_3$ are independent, then $\text{cov}(X_2, X_1 + X_3) = \text{cov}(X_2, X_1) + \text{cov}(X_2, X_3) = 0$ and so

$$\text{cov}(X_2, X_1) = -\text{cov}(X_2, X_3). \quad (2)$$

Finally, if X_3 and $X_1 + X_2$ are independent, then $\text{cov}(X_3, X_1 + X_2) = \text{cov}(X_3, X_1) + \text{cov}(X_3, X_2) = 0$ and so

$$\text{cov}(X_3, X_1) = -\text{cov}(X_3, X_2). \quad (3)$$

Since (1), (2), and (3) must be simultaneously satisfied, the only possibility is that $\text{cov}(X_1, X_2) = \text{cov}(X_1, X_3) = \text{cov}(X_2, X_3) = 0$. Hence, X_1 , X_2 , and X_3 are independent as required.

Problem #12. Using Theorem 5.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix} \right)$$

and so we see that $Y_1 \in N(2, 10)$, $Y_2 \in N(-1, 5)$, and $\text{corr}(Y_1, Y_2) = \frac{1}{\sqrt{2}}$. Thus, by the results in Section 5.6, the distribution of $Y_1|Y_2 = y$ is normal with mean $2 + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{10}}{\sqrt{5}}(y - (-1)) = y + 3$ and variance $10 \left(1 - \left(\frac{1}{\sqrt{2}} \right)^2 \right) = 5$. That is,

$$Y_1|Y_2 = y \in N(y + 3, 5).$$

Problem #13. Let $\mathbf{X} = (X_1, X_2, X_3)'$ where X_1, X_2, X_3 are i.i.d. $N(1, 1)$ so that $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{Y} = (U, V)'$ where $U = 2X_1 - X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$. If

$$B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

then $\mathbf{Y} = B\mathbf{X}$. By Theorem 5.3.1, \mathbf{Y} is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

and covariance matrix

$$B\Lambda B' = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 14 \end{pmatrix}.$$

We can immediately conclude that $U \in N(2, 6)$, $V \in N(6, 14)$, and $\text{cov}(U, V) = 3$ so that $\text{corr}(U, V) = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}}$. It follows from Section 5.6 that the distribution of $V|U = u$ is

$$N\left(6 + \frac{3}{2\sqrt{21}} \frac{\sqrt{14}}{\sqrt{6}}(u - 2), 14 \left(1 - \frac{9}{4 \cdot 21}\right)\right).$$

Choosing $u = 3$ therefore implies that

$$V|U = 3 \in N(6.5, 12.5).$$

Problem #15. Using Theorem 5.3.1, the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$ is

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 & -5 \\ 4 & 9 & -10 \\ -5 & -10 & 13 \end{pmatrix}\right)$$

and so we see that $X_1 \in N(0, 2)$, $X_2 \in N(0, 9)$, and $X_3 \in N(0, 13)$. Since $\text{cov}(X_1, X_3) = -5$, we conclude that $X_1 + X_3 \in N(0, 5)$. Finally, we compute $\text{cov}(X_2, X_1 + X_3) = \text{cov}(X_2, X_1) + \text{cov}(X_2, X_3) = 4 - 10 = -6$ and so $\text{corr}(X_2, X_1 + X_3) = -\frac{2}{\sqrt{5}}$. Thus, by the results in Section 5.6, the distribution of $X_2|X_1 + X_3 = x$ is normal with mean $0 - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{5}}(x - 0) = -\frac{6x}{5}$ and variance $9 \left(1 - \left(-\frac{2}{\sqrt{5}}\right)^2\right) = \frac{9}{5}$. That is,

$$X_2|X_1 + X_3 = x \in N\left(-\frac{6x}{5}, \frac{9}{5}\right).$$

Problem #16. Using Theorem 5.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right).$$

By definition,

$$f_{Y_1|Y_2=0, Y_3=0}(y) = \frac{f_{Y_1, Y_2, Y_3}(y, 0, 0)}{f_{Y_2, Y_3}(0, 0)}.$$

From Definition III, we know

$$f_{Y_1, Y_2, Y_3}(y, 0, 0) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}} e^{-\frac{1}{2} \frac{3}{4} y^2}$$

since

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

The joint distribution of $(Y_2, Y_3)'$ is

$$(Y_2, Y_3)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)$$

and so

$$f_{Y_2, Y_3}(0, 0) = \frac{1}{2\pi\sqrt{3}}.$$

Thus, we conclude

$$f_{Y_1|Y_2=0, Y_3=0}(y) = \frac{\left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}} e^{-\frac{1}{2} \frac{3}{4} y^2}}{\frac{1}{2\pi\sqrt{3}}} = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{2} \exp\left\{-\frac{1}{2} \left(\frac{y}{2/\sqrt{3}}\right)^2\right\}$$

which we recognize as the density function of a normal random variable with mean 0 and variance $4/3$. That is,

$$Y_1|Y_2 = Y_3 = 0 \in N\left(0, \frac{4}{3}\right).$$

Problem #25. Using Theorem 5.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 9 & 6 \\ 6 & 6 \end{pmatrix}\right)$$

and so we see that $Y_1 \in N(3, 9)$, $Y_2 \in N(2, 6)$, and $\text{corr}(Y_1, Y_2) = \frac{\sqrt{2}}{\sqrt{3}}$. Thus, by the results in Section 5.6, the distribution of $Y_1|Y_2 = 0$ is normal with mean $3 + \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{3}{\sqrt{6}}(0 - 2) = 1$ and variance $9\left(1 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\right) = 3$. That is,

$$Y_1|Y_2 = 0 \in N(1, 3).$$

Problem #39. In order to determine the values of a and b for which $\mathbb{E}(U - a - bV)^2$ is a minimum, we must minimize the function $g(a, b) = \mathbb{E}(U - a - bV)^2$. If $U = X_1 + X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$, then

$$U - a - bV = X_1 + X_2 + X_3 - a - b(X_1 + 2X_2 + 3X_3) = (1 - b)X_1 + (1 - 2b)X_2 + (1 - 3b)X_3 - a.$$

Notice that $\mathbb{E}(U - a - bV)^2 = \text{var}(U - a - bV) + [\mathbb{E}(U - a - bV)]^2$. We now compute

$$\begin{aligned} \text{var}(U - a - bV) &= \text{var}((1 - b)X_1 + (1 - 2b)X_2 + (1 - 3b)X_3 - a) \\ &= (1 - b)^2 \text{var}(X_1) + (1 - 2b)^2 \text{var}(X_2) + (1 - 3b)^2 \text{var}(X_3) \\ &= (1 - b)^2 + (1 - 2b)^2 + (1 - 3b)^2 \end{aligned}$$

using the fact that X_1, X_2, X_3 are i.i.d. $N(1, 1)$. Furthermore,

$$\begin{aligned} \mathbb{E}(U - a - bV) &= \mathbb{E}((1 - b)X_1 + (1 - 2b)X_2 + (1 - 3b)X_3 - a) = (1 - b) + (1 - 2b) + (1 - 3b) - a \\ &= 3 - 6b - a \end{aligned}$$

which implies that

$$g(a, b) = (1 - b)^2 + (1 - 2b)^2 + (1 - 3b)^2 + [3 - 6b - a]^2 = 12 - 48b + 50b^2 - 6a + 12ab + a^2.$$

To minimize g , we begin by finding the critical points. That is,

$$\frac{\partial}{\partial a}g(a, b) = -6 + 12b + 2a = 0$$

implies $a + 6b = 3$, and

$$\frac{\partial}{\partial b}g(a, b) = -48 + 100b + 12a = 0$$

implies $25b + 3a = 12$. Solving the second equation for b yields

$$25b = 12 - 3a = 12 - 3(3 - 6b) \quad \text{and so} \quad b = \frac{3}{7}.$$

Substituting in gives

$$a = 3 - 6b = 3 - \frac{18}{7} = \frac{3}{7}.$$

Since

$$\frac{\partial^2}{\partial a^2}g(a, b) = 2 > 0$$

and

$$\frac{\partial^2}{\partial a^2}g(a, b) \cdot \frac{\partial^2}{\partial b^2}g(a, b) - \left(\frac{\partial^2}{\partial a \partial b}g(a, b) \right)^2 = 2 \cdot 100 - 12^2 = 56 > 0$$

we conclude by the second derivative test that $a = 3/7$, $b = 3/7$ is indeed the minimum.