Stat 351 Fall 2009
Chapter 4 Solutions

Problem #3. If $0 \le y \le 1/2$, then
$$f_Y(y) = \int_y^{1-y} f_{X_{(1)},X_{(2)}}(y,z)\,\mathrm{d}z = \int_y^{1-y} 2\,\mathrm{d}z = 2(1-2y).$$
On the other hand, if $1/2 \le y \le 1$, then
$$f_Y(y) = \int_{1-y}^{y} f_{X_{(1)},X_{(2)}}(y,z)\,\mathrm{d}z = \int_{1-y}^{y} 2\,\mathrm{d}z = 2(2y-1).$$
Problem #5. Since $E[F(X_{(n)}) - F(X_{(1)})] = E[F(X_{(n)})] - E[F(X_{(1)})]$, we compute each of $E[F(X_{(n)})]$ and $E[F(X_{(1)})]$ separately. Therefore, by definition,
$$E[F(X_{(n)})] = \int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,\mathrm{d}y_n.$$
From Theorem 4.1.2, we know that $f_{X_{(n)}}(y_n) = n[F(y_n)]^{n-1} f(y_n)$ so that
$$\int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,\mathrm{d}y_n = \int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,\mathrm{d}y_n.$$
Making the substitution $u = F(y_n)$ so that $\mathrm{d}u = F'(y_n)\,\mathrm{d}y_n = f(y_n)\,\mathrm{d}y_n$ gives
$$\int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,\mathrm{d}y_n = \int_0^1 nu^n\,\mathrm{d}u = \frac{n}{n+1}.$$
Note that since F is a distribution, our new limits of integration are $F(-\infty) = 0$ and $F(\infty) = 1$. As for $E[F(X_{(1)})]$, using Theorem 4.1.2, we compute
$$E[F(X_{(1)})] = \int_{-\infty}^{\infty} F(y_1) f_{X_{(1)}}(y_1)\,\mathrm{d}y_1 = \int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,\mathrm{d}y_1.$$
Making the same substitution as above gives
$$\int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,\mathrm{d}y_1 = \int_0^1 nu(1-u)^{n-1}\,\mathrm{d}u = n\int_0^1 (1-v)v^{n-1}\,\mathrm{d}v = 1 - \frac{n}{n+1}.$$
Finally, we combine our two results to conclude that
$$E[F(X_{(n)}) - F(X_{(1)})] = \frac{n}{n+1} - \left[1 - \frac{n}{n+1}\right] = \frac{n-1}{n+1}.$$
Problem #6. (a) By Theorem 4.3.1, the joint density of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})'$ is
$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4) = 24$$
provided that $0 < y_1 < y_2 < y_3 < y_4 < 1$. Therefore,
$$\begin{aligned} f_{X_{(3)},X_{(4)}}(y_3,y_4) = \int_0^{y_3}\int_0^{y_2} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,\mathrm{d}y_1\,\mathrm{d}y_2 &= \int_0^{y_3}\int_0^{y_2} 24\,\mathrm{d}y_1\,\mathrm{d}y_2 \\ &= \int_0^{y_3} 24y_2\,\mathrm{d}y_2 \\ &= 12y_3^2 \end{aligned}$$
provided that $0 < y_3 < y_4 < 1$.

(continued)

Observe that we must necessarily have $X_{(3)} \le 1/2$ for otherwise we would necessarily have $X_{(3)} + X_{(4)} \ge 1$. We therefore conclude that

$$\begin{aligned} P(X_{(3)} + X_{(4)} \le 1) = \int_0^{1/2} \int_y^{1-y} f_{X_{(3)},X_{(4)}}(y,z)\,\mathrm{d}z\,\mathrm{d}y &= \int_0^{1/2} \int_y^{1-y} 12y^2\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^{1/2} 12y^2(1-2y)\,\mathrm{d}y \\ &= (4y^3 - 6y^4)\Big|_0^{1/2} \\ &= \frac{1}{8}. \end{aligned}$$

(b) By the law of total probability, we have

$$P(X_3 + X_4 \le 1) = \int_0^1 P(X_4 \le 1 - x | X_3 = x) f_{X_3}(x)\,\mathrm{d}x.$$

Since X_3 and X_4 are independent $U(0,1)$ random variables, we find $P(X_4 \le 1 - x|X_3 = x) = P(X_4 \le 1 - x) = 1 - x$. Thus, we conclude

$$P(X_3 + X_4 \le 1) = \int_0^1 (1-x)\,\mathrm{d}x = \frac{1}{2}.$$

Problem #7. By Theorem 4.3.1, the joint density of $(X_{(1)}, X_{(2)}, X_{(3)})'$ is

$$f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3) = 6$$

provided that $0 < y_1 < y_2 < y_3 < 1$. Therefore,

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = \int_{y_1}^{y_3} f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3)\,\mathrm{d}y_2 = \int_{y_1}^{y_3} 6\,\mathrm{d}y_2 = 6(y_3 - y_1)$$

provided that $0 < y_1 < y_3 < 1$. (This is also equation (3.10) on page 112.) Observe that we must necessarily have $X_{(1)} \le 1/2$ for otherwise we would necessarily have $X_{(1)} + X_{(3)} \ge 1$. We therefore conclude

$$\begin{aligned} P(X_{(1)} + X_{(3)} \le 1) = \int_0^{1/2} \int_x^{1-x} f_{X_{(1)},X_{(3)}}(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{1/2} \int_x^{1-x} 6(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{1/2} (3y^2 - 6xy)\Big|_{y=x}^{y=1-x} \mathrm{d}x \\ &= \int_0^{1/2} 12x^2 - 12x + 3\,\mathrm{d}x \\ &= (4x^3 - 6x^2 + 3x)\Big|_0^{1/2} \\ &= \frac{1}{2}. \end{aligned}$$

Problem #8. By Theorem 4.3.1, the joint density of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})'$ is

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4) = 24$$

provided that $0 < y_1 < y_2 < y_3 < y_4 < 1$.

(continued)

(a) We find

$$\begin{aligned} f_{X_{(2)},X_{(3)}}(y_2,y_3) = \int_0^{y_2}\int_{y_3}^1 f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,\mathrm{d}y_4\,\mathrm{d}y_1 &= \int_0^{y_2}\int_{y_3}^1 24\,\mathrm{d}y_4\,\mathrm{d}y_1 \\ &= \int_0^{y_2} 24(1-y_3)\,\mathrm{d}y_1 \\ &= 24y_2(1-y_3) \end{aligned}$$

provided that $0 < y_2 < y_3 < 1$. Observe that we must necessarily have $X_{(2)} \le 1/2$ for otherwise we would necessarily have $X_{(2)} + X_{(3)} \ge 1$. We therefore conclude

$$\begin{aligned} P(X_{(2)} + X_{(3)} \le 1) = \int_0^{1/2}\int_x^{1-x} f_{X_{(2)},X_{(3)}}(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{1/2}\int_x^{1-x} 24x(1-y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{1/2} (24xy - 12xy^2)\Big|_{y=x}^{y=1-x}\,\mathrm{d}x \\ &= \int_0^{1/2} 12x(1-2x)\,\mathrm{d}x \\ &= (6x^2 - 8x^3)\Big|_{x=0}^{x=1/2} \\ &= \frac{1}{2}. \end{aligned}$$

(b) We find

$$\begin{aligned} f_{X_{(1)},X_{(2)}}(y_1,y_2) = \int_{y_2}^1\int_{y_2}^{y_4} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,\mathrm{d}y_3\,\mathrm{d}y_4 &= \int_{y_2}^1\int_{y_2}^{y_4} 24\,\mathrm{d}y_3\,\mathrm{d}y_4 \\ &= \int_{y_2}^1 24(y_4-y_2)\,\mathrm{d}y_4 \\ &= (12y_4^2 - 24y_2y_4)\Big|_{y_4=y_2}^{y_4=1} \\ &= 12(1-y_2)^2 \end{aligned}$$

provided that $0 < y_1 < y_2 < 1$, and so

$$\begin{aligned} P(X_{(2)} \le 3X_{(1)}) = \int_0^1\int_{y/3}^{y} f_{X_{(1)},X_{(2)}}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_{y/3}^{y} 12(1-y)^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 8y(1-y)^2\,\mathrm{d}y \\ &= \left[4y^2 - \frac{16}{3}y^3 + 2y^4\right]_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

Problem #9. By Theorem 4.3.1, the joint density of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})'$ is

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4) = 24$$

provided that $0 < y_1 < y_2 < y_3 < y_4 < 1$.

(continued)

Therefore,

$$\begin{aligned} f_{X_{(1)},X_{(3)}}(y_1,y_3) &= \int_{y_3}^{1}\int_{y_1}^{y_3} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,\mathrm{d}y_2\,\mathrm{d}y_4 \\ &= \int_{y_3}^{1}\int_{y_1}^{y_3} 24\,\mathrm{d}y_2\,\mathrm{d}y_4 \\ &= 24(y_3-y_1)(1-y_3) \end{aligned}$$

provided that $0 < y_1 < y_3 < 1$, and

$$\begin{aligned} f_{X_{(2)},X_{(4)}}(y_2,y_4) &= \int_{0}^{y_2}\int_{y_2}^{y_4} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,\mathrm{d}y_3\,\mathrm{d}y_1 \\ &= \int_{0}^{y_2}\int_{y_2}^{y_4} 24\,\mathrm{d}y_3\,\mathrm{d}y_1 \\ &= 24y_2(y_4-y_2) \end{aligned}$$

provided that $0 < y_2 < y_4 < 1$.

(a) Let $U = X_{(3)} - X_{(1)}$ and $V = X_{(1)}$ so that solving for $X_{(1)}$ and $X_{(3)}$ gives

$$X_{(1)} = V \quad \text{and} \quad X_{(3)} = U + V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial y_1}{\partial u} & \dfrac{\partial y_1}{\partial v} \\ \dfrac{\partial y_3}{\partial u} & \dfrac{\partial y_3}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X_{(1)},X_{(3)}}(v,u+v)\cdot|J| = 24u(1-u-v)$$

provided that $0 < u < 1-v$ and $0 < v < 1$, or equivalently, $0 < v < 1-u$ and $0 < u < 1$. Thus, we conclude that the density of $U = X_{(3)} - X_{(1)}$ is

$$f_U(u) = \int_0^{1-u} 24u(1-u-v)\,\mathrm{d}v = \left(24u(1-u)v - 12uv^2\right)\Big|_{v=0}^{v=1-u} = 12u(1-u)^2$$

for $0 < u < 1$.

(b) Let $U = X_{(4)} - X_{(2)}$ and $V = X_{(2)}$ so that solving for $X_{(2)}$ and $X_{(4)}$ gives

$$X_{(2)} = V \quad \text{and} \quad X_{(4)} = U + V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial y_2}{\partial u} & \dfrac{\partial y_2}{\partial v} \\ \dfrac{\partial y_4}{\partial u} & \dfrac{\partial y_4}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

(continued)

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X_{(2)},X_{(4)}}(v, u+v) \cdot |J| = 24uv$$

provided that $0 < u < 1 - v$ and $0 < v < 1$, or equivalently, $0 < v < 1 - u$ and $0 < u < 1$. Thus, we conclude that the density of $U = X_{(4)} - X_{(2)}$ is

$$f_U(u) = \int_0^{1-u} 24uv \, \mathrm{d}v = 24uv^2 \Big|_{v=0}^{v=1-u} = 12u(1-u)^2$$

for $0 < u < 1$.

Problem #10. This is an extremely tricky problem. We begin by noting that by Theorem 4.3.1, the joint density of $(X_{(1)}, X_{(2)}, X_{(3)})'$ is $f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3) = 6$ provided that $0 < y_1 < y_2 < y_3 < 1$. We also observe that

$$P(X_{(1)} + X_{(2)} > X_{(3)}) = \iint\limits_{\left\{\substack{0<x+y<2,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < \min\{x+y, 1\},\, X_{(1)} = x,\, X_{(2)} = y) \, \mathrm{d}x \, \mathrm{d}y$$

where the upper limit for $X_{(3)}$ follows since $X_{(3)}$ is necessarily less than 1 although $X_{(1)}+X_{(2)} = x+y$ could be greater than 1, and the constraint that $0 < x + y < 2$ follows since each of x and y are between 0 and 1. We now write

$$\begin{aligned}
&\iint\limits_{\left\{\substack{0<x+y<2,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < \min\{x+y, 1\},\, X_{(1)} = x,\, X_{(2)} = y) \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad = \iint\limits_{\left\{\substack{0<x+y<1,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < x+y,\, X_{(1)} = x,\, X_{(2)} = y) \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad\qquad + \iint\limits_{\left\{\substack{1<x+y<2,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < 1,\, X_{(1)} = x,\, X_{(2)} = y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

We now find

$$\begin{aligned}
\iint\limits_{\left\{\substack{0<x+y<1,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < x+y\},\, X_{(1)} = x,\, X_{(2)} = y) \, \mathrm{d}x \, \mathrm{d}y &= \int_0^{1/2} \int_x^{1-x} \int_y^{x+y} f_{X_{(1)},X_{(2)},X_{(3)}}(x,y,z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^{1/2} \int_x^{1-x} \int_y^{x+y} 6 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^{1/2} \int_x^{1-x} 6x \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^{1/2} 6x(1-2x) \, \mathrm{d}x \\
&= (3x^2 - 4x^3) \Big|_0^{1/2} \\
&= \frac{1}{4}.
\end{aligned}$$

Note that you should draw the region $\{0 < x + y < 1,\, 0 < x < y < 1\}$ to ensure you understand the limits of integration. *(continued)*

We also find

$$\iint_{\left\{\substack{1<x+y<2,\\ 0<x<y<1}\right\}} P(y < X_{(3)} < 1,\, X_{(1)} = x,\, X_{(2)} = y)\,\mathrm{d}x\,\mathrm{d}y = \int_{1/2}^{1}\int_{1-y}^{y}\int_{y}^{1} f_{X_{(1)},X_{(2)},X_{(3)}}(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{1/2}^{1}\int_{1-y}^{y}\int_{y}^{1} 6\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{1/2}^{1}\int_{1-y}^{y} 6(1-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{1/2}^{1} 6(1-y)(2y-1)\,\mathrm{d}y$$

$$= (9y^2 - 6y - 4y^3)\Big|_{1/2}^{1}$$

$$= \frac{1}{4}.$$

Note that you should draw the region $\{1 < x + y < 2,\, 0 < x < y < 1\}$ to ensure you understand the limits of integration. Therefore, we finally conclude that

$$P(X_{(1)} + X_{(2)} > X_{(3)}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Problem #11. If X_1, X_2, X_3 are i.i.d. $U(0,1)$ random variables, then we know from Problem #7 (or equation (3.10) on page 112) that

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = 6(y_3 - y_1)$$

provided $0 < y_1 < y_3 < 1$. Since $X_{(3)} > X_{(1)}$ we immediately see that $P(X_{(3)} > aX_{(1)}) = 1$ for any $a \le 1$. Thus, we will now compute $P(X_{(3)} > aX_{(1)})$ for any $a > 1$. Observe that if $X_{(1)} > 1/a$, then it is not possible for $X_{(3)}$ to be larger than $aX_{(1)}$. Thus,

$$\begin{aligned} P(X_{(3)} > aX_{(1)}) &= P(X_{(3)} > aX_{(1)},\, X_{(1)} > 1/a) + P(X_{(3)} > aX_{(1)},\, X_{(1)} \le 1/a) \\ &= P(X_{(3)} > aX_{(1)},\, X_{(1)} \le 1/a). \end{aligned}$$

We now observe that

$$P(X_{(3)} > aX_{(1)},\, X_{(1)} \le 1/a) = \int_0^{1/a}\int_{ax}^{1} f_{X_{(1)},X_{(3)}}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{1/a}\int_{ax}^{1} 6(y-x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{1/a} (3y^2 - 6xy)\Big|_{y=ax}^{y=1}\,\mathrm{d}x$$

$$= \int_0^{1/a} [3(2a - a^2)x^2 - 6x + 3]\,\mathrm{d}x$$

$$= \left[(2a - a^2)x^3 - 3x^2 + 3x\right]\Big|_0^{1/a}$$

$$= \frac{2a-1}{a^2}.$$

(a) Thus, if $a = 2$, then the required probability is $3/4$.

(continued)

(b) If we want to find the value of a such that the required probability is 1/2, then we simply solve

$$\frac{2a-1}{a^2} = \frac{1}{2}$$

for a. Doing so implies a satisfies $a^2 - 4a + 2 = 0$ so that

$$a = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

Since $a > 1$, we conclude that $a = 2 + \sqrt{2}$.

Problem #15. By definition,

$$\rho_{X_{(1)},X_{(3)}} = \frac{\text{cov}(X_{(1)}, X_{(3)})}{\sqrt{\text{var}(X_{(1)}) \cdot \text{var}(X_{(3)})}}.$$

Since X_1, X_2, X_3 are i.i.d. Exp(1) random variables, we conclude from Theorem 4.2.1 that

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = 6(e^{-y_1} - e^{-y_3})e^{-y_1}e^{-y_3}$$

provided $0 < y_1 < y_3 < \infty$. We also conclude from Theorem 4.1.2 that

$$f_{X_{(1)}}(y_1) = 3(e^{-y_1})^2 e^{-y_1} = 3e^{-3y_1}$$

provided that $0 < y_1 < \infty$, and that

$$f_{X_{(3)}}(y_3) = 3(1 - e^{-y_3})^2 e^{-y_3}$$

provided that $0 < y_3 < \infty$. Since we recognize $X_{(1)} \in \text{Exp}(1/3)$ we conclude immediately that $E(X_{(1)}) = 1/3$ and $\text{var}(X_{(1)}) = 1/9$. Next we compute

$$\begin{aligned} E(X_{(3)}) = \int_0^\infty 3y_3(1 - e^{-y_3})^2 e^{-y_3} \, \mathrm{d}y_3 &= \int_0^\infty 3y_3 e^{-y_3} \, \mathrm{d}y_3 - \int_0^\infty 6y_3 e^{-2y_3} \, \mathrm{d}y_3 + \int_0^\infty 3y_3 e^{-3y_3} \, \mathrm{d}y_3 \\ &= 3\Gamma(2) - 6\left(\frac{1}{2}\right)^2 \Gamma(2) + 3\left(\frac{1}{3}\right)^2 \Gamma(2) \\ &= \frac{11}{6} \end{aligned}$$

and

$$\begin{aligned} E(X_{(3)}^2) = \int_0^\infty 3y_3^2(1 - e^{-y_3})^2 e^{-y_3} \, \mathrm{d}y_3 &= \int_0^\infty 3y_3^2 e^{-y_3} \, \mathrm{d}y_3 - \int_0^\infty 6y_3^2 e^{-2y_3} \, \mathrm{d}y_3 + \int_0^\infty 3y_3^2 e^{-3y_3} \, \mathrm{d}y_3 \\ &= 3\Gamma(3) - 6\left(\frac{1}{2}\right)^3 \Gamma(3) + 3\left(\frac{1}{3}\right)^3 \Gamma(3) \\ &= \frac{85}{18}. \end{aligned}$$

Therefore,

$$\text{var}(X_{(3)}) = E(X_{(3)}^2) - [E(X_{(3)})]^2 = \frac{85}{18} - \left(\frac{11}{6}\right)^2 = \frac{49}{36}.$$

(continued)

Now we compute

$$
\begin{aligned}
E(X_{(1)}X_{(3)}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_{(1)},X_{(3)}}(y_1,y_3)\,\mathrm{d}y_3\,\mathrm{d}y_1 = \int_0^{\infty}\int_{y_1}^{\infty} 6y_1y_3(e^{-y_1}-e^{-y_3})e^{-y_1}e^{-y_3}\,\mathrm{d}y_3\,\mathrm{d}y_1 \\
&= 6\int_0^{\infty} y_1e^{-y_1}\int_{y_1}^{\infty} y_3(e^{-y_1}-e^{-y_3})e^{-y_3}\,\mathrm{d}y_3\,\mathrm{d}y_1. \\
&= 6\int_0^{\infty} y_1e^{-2y_1}\int_{y_1}^{\infty} y_3e^{-y_3}\,\mathrm{d}y_3\,\mathrm{d}y_1 - 6\int_0^{\infty} y_1e^{-y_1}\int_{y_1}^{\infty} y_3e^{-2y_3}\,\mathrm{d}y_3\,\mathrm{d}y_1 \\
&= 6\int_0^{\infty} y_1e^{-2y_1}(y_1e^{-y_1}+e^{-y_1})\,\mathrm{d}y_1 - 6\int_0^{\infty} y_1e^{-y_1}\left(\frac{1}{2}y_1e^{-2y_1}+\frac{1}{4}e^{-2y_1}\right)\mathrm{d}y_1 \\
&= 3\int_0^{\infty} y_1^2e^{-3y_1}\,\mathrm{d}y_1 + \frac{9}{2}\int_0^{\infty} y_1e^{-3y_1}\,\mathrm{d}y_1 \\
&= 3\Gamma(3)\left(\frac{1}{3}\right)^3 + \frac{9}{2}\Gamma(2)\left(\frac{1}{3}\right)^2 \\
&= \frac{13}{18}
\end{aligned}
$$

so that

$$\operatorname{cov}(X_{(1)},X_{(3)}) = E(X_{(1)}X_{(3)}) - E(X_{(1)})E(X_{(3)}) = \frac{13}{18} - \frac{1}{3}\cdot\frac{11}{6} = \frac{1}{9}.$$

Finally, we put everything together to conclude

$$\rho_{X_{(1)},X_{(3)}} = \frac{\operatorname{cov}(X_{(1)},X_{(3)})}{\sqrt{\operatorname{var}(X_{(1)})\cdot\operatorname{var}(X_{(3)})}} = \frac{1/9}{1/3\cdot 7/6} = \frac{2}{7}.$$

Problem #16. (a) If X_1 and X_2 are independent Exp(a) random variables, then by Theorem 4.2.1, the joint density of $(X_{(1)},X_{(2)})'$ is given by

$$f_{X_{(1)},X_{(2)}}(y_1,y_2) = \frac{2}{a^2}\exp\left(-\frac{y_1+y_2}{a}\right)$$

provided that $0<y_1<y_2<\infty$. Suppose that $U=X_{(1)}$ and let $V=X_{(2)}-X_{(1)}$. Solving for $X_{(1)}$ and $X_{(2)}$ gives

$$X_{(1)} = U \quad\text{and}\quad X_{(2)} = U+V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial y_1}{\partial u} & \dfrac{\partial y_1}{\partial v} \\ \dfrac{\partial y_2}{\partial u} & \dfrac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Therefore, by Theorem 1.2.1, the density of $(U,V)'$ is given by

$$f_{U,V}(u,v) = f_{X_{(1)},X_{(2)}}(u,u+v)\cdot|J| = \frac{2}{a^2}\exp\left(-\frac{u+u+v}{a}\right) = \frac{2}{a^2}\exp\left(-\frac{2u+v}{a}\right) = \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}$$

provided that $v>0$ and $u>0$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_0^{\infty}\frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}\,\mathrm{d}v = \frac{2}{a}e^{-2u/a}$$

for $u>0$.

(continued)

We recognize that this is the density of an exponential random variable with parameter $a/2$; that is, $U = X_{(1)} \in \text{Exp}(a/2)$. The marginal density of V is

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}u = \int_0^{\infty} \frac{2}{a}e^{-2u/a} \cdot \frac{1}{a}e^{-v/a}\,\mathrm{d}u = \frac{1}{a}e^{-v/a}$$

for $v > 0$. We recognize that this is the density of an exponential random variable with parameter a; that is, $V = X_{(2)} - X_{(1)} \in \text{Exp}(a)$. Since we can express $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$ we conclude that U and V are independent; in other words, $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent.

(b) To compute $E(X_{(2)}|X_{(1)} = y)$, we can use properties of conditional expectation (Theorem 2.2.2):

$$\begin{aligned} E(X_{(2)}|X_{(1)} = y) &= E(X_{(2)} - X_{(1)} + X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}|X_{(1)} = y) + E(X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}) + y \\ &= a + y \end{aligned}$$

where the first expression after the third equality follows since $X_{(2)} - X_{(1)}$ is independent of $X_{(1)}$ and the second expression follows since $X_{(1)}$ is "known" when conditioned on the value $X_{(1)} = y$.

As for $E(X_{(1)}|X_{(2)} = x)$, we need to compute this by definition of conditional expectation. That is,

$$f_{X_{(1)}|X_{(2)}=x}(y_1) = \frac{f_{X_{(1)},X_{(2)}}(y_1,x)}{f_{X_{(2)}}(x)} = \frac{\frac{2}{a^2}e^{-y_1/a} \cdot e^{-x/a}}{\frac{2}{a}(1-e^{-x/a}) \cdot e^{-x/a}} = \frac{1}{a}\frac{e^{-y_1/a}}{1-e^{-x/a}}$$

provided $0 < y_1 < x$. This then gives

$$E(X_{(1)}|X_{(2)} = x) = \int_{-\infty}^{\infty} f_{X_{(1)}|X_{(2)}=x}(y_1)\,\mathrm{d}y_1 = \int_0^x \frac{y_1}{a}\frac{e^{-y_1/a}}{1-e^{-x/a}}\,\mathrm{d}y_1 = \frac{1}{a(1-e^{-x/a})}\int_0^x y_1\,e^{-y_1/a}\,\mathrm{d}y_1.$$

Integrating by parts gives

$$\int_0^x y_1\,e^{-y_1/a}\,\mathrm{d}y_1 = a^2 - a^2e^{-x/a} - axe^{-x/a}.$$

Therefore,

$$E(X_{(1)}|X_{(2)} = x) = \frac{a^2 - a^2e^{-x/a} - axe^{-x/a}}{a(1-e^{-x/a})} = a - \frac{xe^{-x/a}}{1-e^{-x/a}} = a - \frac{x}{e^{x/a}-1}.$$

Problem #17. Let X_1, X_2, and X_3 are independent, identically distributed $U(0,1)$ random variables. Notice that if $x > 1/2$, then since $X_{(3)} > X_{(1)}$ we conclude

$$P(X_{(3)} > \tfrac{1}{2}\,|\,X_{(1)} = x) = 1.$$

On the other hand, suppose that $0 \le x \le 1/2$. As in Problem #7 (or by equation (3.10) on page 112),

$$f_{X_{(1)},X_{(3)}}(y_1,y_3) = 6(y_3 - y_1)$$

provided $0 < y_1 < y_3 < 1$.

(continued)

Therefore, we find

$$P(X_{(3)} > \tfrac{1}{2} \mid X_{(1)} = x) = \frac{\displaystyle\int_{1/2}^{1} f_{X_{(1)},X_{(3)}}(x, y_3)\,\mathrm{d}y_3}{f_{X_{(1)}}(x)}.$$

For the numerator we calculate

$$\int_{1/2}^{1} f_{X_{(1)},X_{(3)}}(x, y_3)\,\mathrm{d}y_3 = \int_{1/2}^{1} 6(y_3 - x)\,\mathrm{d}y_3 = (3y_3^2 - 6xy_3)\Big|_{1/2}^{1} = \frac{9}{4} - 3x = \frac{3}{4}(3 - 4x).$$

As for the denominator, by Theorem 4.1.2, we find

$$f_{X_{(1)}}(x) = 3(1 - x)^2$$

provided $0 < x < 1$. Putting these pieces together, we conclude

$$P(X_{(3)} > \tfrac{1}{2} \mid X_{(1)} = x) = \frac{\frac{3}{4}(3 - 4x)}{3(1 - x)^2} = \frac{(3 - 4x)}{4(1 - x)^2}.$$

That is,

$$P(X_{(3)} > \tfrac{1}{2} \mid X_{(1)} = x) = \begin{cases} \frac{(3-4x)}{4(1-x)^2}, & \text{if } 0 \le x \le 1/2, \\ 1, & \text{if } x > 1/2. \end{cases}$$

Problem #19. Since $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are i.i.d. $U(0, a)$ random variables, we can immediately conclude that $X_{(n)}$ and $Y_{(n)}$ are independent and identically distributed. Furthermore, we can use Theorems 4.1.1 and 4.1.2 to determine their common distribution and density functions. That is, $X_{(n)}$ and $Y_{(n)}$ have common distribution function

$$F(x) = \begin{cases} 0, & x \le 0, \\ \frac{x^n}{a^n}, & 0 < x < a, \\ 1, & x \ge 1, \end{cases}$$

and common density function

$$f(x) = \frac{n}{a^n} x^{n-1}, \quad 0 < x < a.$$

If we now let $S = \min\{X_{(n)}, Y_{(n)}\}$ and $T = \max\{X_{(n)}, Y_{(n)}\}$, then Theorem 4.2.1 implies that the joint density of $(S, T)'$ is

$$f_{S,T}(s, t) = 2 \cdot \frac{n}{a^n} s^{n-1} \cdot \frac{n}{a^n} t^{n-1} = \frac{2n^2}{a^{2n}} s^{n-1} t^{n-1}, \quad 0 < s < t < a.$$

The next step is to let $U = \frac{T}{S}$ and $V = S$ so that $S = V$ and $T = UV$. We find the Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial s}{\partial u} & \dfrac{\partial s}{\partial v} \\ \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v.$$

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u, v) = f_{S,T}(v, uv) \cdot |J| = \frac{2n^2}{a^{2n}} v^{n-1} (uv)^{n-1} \cdot v = \frac{2n^2}{a^{2n}} u^{n-1} v^{2n-1}$$

provided that $1 < u < \infty$, $0 < v < \frac{a}{u} < a$.

(continued)

The marginal density for U is therefore given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \frac{2n^2}{a^{2n}}u^{n-1}\int_0^{a/u} v^{2n-1}\,\mathrm{d}v = \frac{n}{a^{2n}}u^{n-1}v^{2n}\Big|_{v=0}^{v=a/u} = \frac{n}{a^{2n}}u^{n-1}\frac{a^{2n}}{u^{2n}} = nu^{-(n+1)}$$

provided that $1 < u < \infty$. Since we are interested in

$$Z_n = n\log\left(\frac{\max\{X_{(n)}, Y_{(n)}\}}{\min\{X_{(n)}, Y_{(n)}\}}\right) = n\log U$$

we can now use techniques from Chapter 1 to find the density of Z_n. Let $Z = Z_n = n\log U$. Therefore, $F_Z(z) = P(Z \le z) = P(U \le e^{z/n})$ and so

$$f_Z(z) = \frac{1}{n}e^{z/n} f_U(e^{z/n}) = \frac{1}{n}e^{z/n}\cdot n(e^{z/n})^{-(n+1)} = e^{-z}$$

provided that $0 < z < \infty$. Hence we conclude that $Z_n \in \mathrm{Exp}(1)$.

Problem #20. **(a)** If $Y_1 = X_{(1)}$ and $Y_k = X_{(k)} - X_{(k-1)}$, $k = 2,\ldots,n$, then solving for $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ gives

$$X_{(1)} = Y_1 \quad \text{and} \quad X_{(k)} = Y_1 + \cdots + Y_k,\ k = 2,\ldots,n.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = 1.$$

(Since the matrix is lower triangular, the determinant is simply the product of the diagonal entries.) By Theorem 1.2.1, we have

$$f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = f_{X_{(1)},\ldots,X_{(n)}}(y_1, y_1 + y_2, \ldots, y_1 + y_2 + \cdots + y_n).$$

Since $X_1, \ldots, X_n$ are $\mathrm{Exp}(a)$ random variables so that they have common density $f(x) = \frac{1}{a}e^{-x/a}$, $x > 0$, we find from Theorem 4.3.1 that the joint density of the order statistic is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n!\prod_{i=1}^n f(x_i) = n!\prod_{i=1}^n \frac{1}{a}e^{-x_i/a} = \frac{n!}{a^n}\exp\left\{-\frac{1}{a}\sum_{i=1}^n x_i\right\}$$

provided $0 < x_1 < x_2 < \cdots < x_n$. Hence, we conclude

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) &= f_{X_{(1)},\ldots,X_{(n)}}(y_1, y_1 + y_2, \ldots, y_1 + y_2 + \cdots + y_n \\ &= \frac{n!}{a^n}\exp\left\{-\frac{1}{a}(ny_1 + (n-1)y_2 + \cdots + 2y_{n-1} + y_n)\right\} \\ &= \frac{n}{a}e^{-ny_1/a}\cdot\frac{(n-1)}{a}e^{-(n-1)y_2/a}\cdots\frac{2}{a}e^{-2y_{n-1}/a}\cdot\frac{1}{a}e^{-y_n/a} \end{aligned}$$

provided that $0 < y_1 < y_1 + y_2 < \cdots < y_1 + \cdots + y_n$, or equivalently, $y_1 > 0$, $y_2 > 0, \ldots, y_n > 0$.

(continued)

In other words, the density function of Y_k is

$$f_{Y_k}(y_k) = \frac{(n+1-k)}{a} \exp\left\{-\frac{(n+1-k)y_k}{a}\right\}, \quad y_k > 0,$$

so that

$$Y_k \in \operatorname{Exp}\left(\frac{a}{n+1-k}\right).$$

(b) Note that

$$Y_1 + Y_2 + \cdots + Y_n = X_{(1)} + \big(X_{(2)} - X_{(1)}\big) + \cdots + \big(X_{(n)} - X_{(n-1)}\big) = X_{(n)}$$

as in **(a)**. Therefore, since $Y_k \in \operatorname{Exp}(\frac{a}{n+1-k})$, we conclude

$$\mathbb{E}(X_{(n)}) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \cdots + \mathbb{E}(Y_n) = \frac{a}{n+1-1} + \frac{a}{n+1-2} + \cdots + \frac{a}{n+1-n} = a\sum_{k=1}^{n} \frac{1}{k}$$

and

$$\operatorname{var}(X_{(k)}) = \operatorname{var}(Y_1) + \operatorname{var}(Y_2) + \cdots + \operatorname{var}(Y_k) = \frac{a^2}{(n+1-1)^2} + \frac{a^2}{(n+1-2)^2} + \cdots + \frac{a^2}{(n+1-n)^2} = a^2 \sum_{k=1}^{n} \frac{1}{k^2}.$$

Problem #21. (a) This is identical to Problem #20(a). Hence, $Y_k \in \operatorname{Exp}(\frac{1}{n+1-k})$ for $k = 1, 2, \ldots, n$.

(b) As in Problem #20(b), we see that $Y_1 + Y_2 + \cdots + Y_n = X_{(n)}$ and so

$$\mathbb{E}(X_{(n)}) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

However, we can also find $\mathbb{E}(X_{(n)})$ another way. By Theorem 4.1.2, we know that

$$f_{X_{(n)}}(x) = n(1 - e^{-x})^{n-1} e^{-x}, \quad 0 < x < \infty$$

and so

$$\mathbb{E}(X_{(n)}) = \int_{-\infty}^{\infty} x f_{X_{(n)}}(x)\,\mathrm{d}x = \int_{0}^{\infty} nx(1 - e^{-x})^{n-1} e^{-x}\,\mathrm{d}x.$$

Equating these two expressions for $\mathbb{E}(X_{(n)})$ gives

$$\int_{0}^{\infty} nx(1 - e^{-x})^{n-1} e^{-x}\,\mathrm{d}x = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

as required.

Problem #22. As in Problem #20(b) we see that

$$X_{(k)} = Y_1 + Y_2 + \cdots + Y_k, \quad k = 1, 2, \ldots, n,$$

where $Y_j \in \operatorname{Exp}(\frac{1}{n+1-j})$ with $Y_1, \ldots, Y_n$ independent. This implies that

$$\begin{aligned} Z_n &= nX_{(1)} + (n-1)X_{(2)} + \cdots + 2X_{(n-1)} + X_{(n)} \\ &= nY_1 + (n-1)(Y_1 + Y_2) + \cdots + 2(Y_1 + Y_2 + \cdots + Y_{n-1}) + (Y_1 + Y_2 + \cdots + Y_n) \\ &= Y_1(1 + 2 + \cdots + n) + Y_2(1 + 2 + \cdots + n - 1) + \cdots + Y_{n-1}(1 + 2) + Y_n. \end{aligned}$$

(continued)

Therefore,

$$\begin{aligned}
\mathbb{E}(Z_n) &= (1+2+\cdots+n)\mathbb{E}(Y_1) + (1+2+\cdots+n-1)\mathbb{E}(Y_2) + \cdots + (1+2)\mathbb{E}(Y_{n-1}) + \mathbb{E}(Y_n) \\
&= (1+2+\cdots+n)\cdot\frac{1}{n+1-1} + (1+2+\cdots+n-1)\cdot\frac{1}{n+1-2} + \cdots + (1+2)\cdot\frac{1}{n+1-(n-1)} + 1 \\
&= \frac{n(n+1)}{2}\cdot\frac{1}{n} + \frac{(n-1)n}{2}\cdot\frac{1}{n-1} + \cdots + \frac{2(3)}{2}\cdot\frac{1}{2} + 1 \\
&= \frac{n+1}{2} + \frac{n}{2} + \cdots + \frac{3}{2} + \frac{2}{2} \\
&= \sum_{k=1}^{n} \frac{k+1}{2} \\
&= \frac{n(n+1)}{4} + \frac{n}{2} \\
&= \frac{n(n+3)}{4}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\operatorname{var}(Z_n) &= (1+2+\cdots+n)^2\operatorname{var}(Y_1) + (1+2+\cdots+n-1)^2\operatorname{var}(Y_2) + \cdots + (1+2)^2\operatorname{var}(Y_{n-1}) + \operatorname{var}(Y_n) \\
&= \left(\frac{n(n+1)}{2}\cdot\frac{1}{n}\right)^2 + \left(\frac{(n-1)n}{2}\cdot\frac{1}{n-1}\right)^2 + \cdots + \left(\frac{2(3)}{2}\cdot\frac{1}{2}\right)^2 + 1 \\
&= \sum_{k=1}^{n} \frac{(k+1)^2}{4} \\
&= \frac{1}{4}\left[\sum_{k=1}^{n+1} k^2\right] - \frac{1}{4} \\
&= \frac{1}{4}\left[\frac{(n+1)(n+2)(2n+3)}{6}\right] - \frac{1}{4} \\
&= \frac{n(2n^2+9n+13)}{24}.
\end{aligned}$$

Problem #24. The key observation is that

$$X_1 + X_2 + \cdots + X_n = X_{(1)} + X_{(2)} + \cdots + X_{(n)}.$$

Using the fact from Stat 251 that the sum of n independent and identically distributed $\operatorname{Exp}(a)$ random variables has a gamma distribution with parameters n and a, we conclude that

$$\sum_{i=1}^{n} X_{(i)} \in \Gamma(n, a).$$

Problem #27. Since $X_1, X_2, \ldots$ are i.i.d. $U(0,1)$ random variables, they have common distribution function

$$F(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

(continued)

Thus, if we let $X_{(n)} = \max\{X_1, \ldots, X_n\}$, then by Theorem 4.1.1, the distribution function of $X_{(n)}$ is

$$F_{X_{(n)}}(y) = \begin{cases} 0, & y \le 0, \\ y^n, & 0 < y < 1, \\ 1, & y \ge 1. \end{cases}$$

Now let $V = \max\{X_1, \ldots, X_N\}$ where $N \in \text{Po}(\lambda)$ is independent of $X_1, X_2, \ldots$. If we condition on the value of N, then there are two cases to consider. Either $N = 0$ which happens with probability $P(N = 0) = e^{-\lambda}$ and so

$$P(V = 0) = P(N = 0) = e^{-\lambda},$$

or $N \ge 1$ in which case the distribution function of $V|N = n$, $n = 1, 2, 3, \ldots$, is given by

$$F_{V|N=n}(v) = \begin{cases} 0, & v \le 0, \\ v^n, & 0 < v < 1, \\ 1, & v \ge 1. \end{cases}$$

Thus, the density function of $V|N = n$, $n = 1, 2, 3, \ldots$, is given by

$$f_{V|N=n}(v) = nv^{n-1}, \quad 0 < v < 1.$$

Finally, we conclude using the law of total probability that the (unconditional) density of V (in the case $N \ge 1$) is

$$\begin{aligned} f_V(v) = \sum_{n=1}^{\infty} f_{V|N=n}(v) P(N = n) = \sum_{n=1}^{\infty} nv^{n-1} \cdot \frac{\lambda^n e^{-\lambda}}{n!} = \frac{e^{-\lambda}}{v} \sum_{n=1}^{\infty} \frac{(\lambda v)^n}{(n-1)!} &= \frac{e^{-\lambda}}{v} \cdot \lambda v \sum_{n=1}^{\infty} \frac{(\lambda v)^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda} e^{\lambda v} \\ &= \lambda e^{-\lambda(1-v)}. \end{aligned}$$

To summarize, we have

- $P(V = 0) = e^{-\lambda}$, and
- $f_V(v) = \lambda e^{-\lambda(1-v)}$, $0 < v < 1$.

Notice that

$$P(V = 0) + \int_0^1 f_V(v)dv = e^{-\lambda} + \int_0^1 \lambda e^{-\lambda(1-v)} dv = e^{-\lambda} + e^{-\lambda}(e^{\lambda} - 1) = 1$$

as expected. Note that V is an example of a random variable which is neither continuous nor discrete. The expected value of V is given by

$$\begin{aligned} \mathbb{E}(V) = 0 \cdot P(V = 0) + \int_0^1 v f_V(v)dv = \int_0^1 \lambda v e^{-\lambda(1-v)} dv &= e^{-\lambda} \int_0^1 \lambda v e^{\lambda v} dv \\ &= \frac{e^{-\lambda}}{\lambda} \int_0^{\lambda} u e^u du \\ &= \frac{e^{-\lambda}}{\lambda} \left[u e^u - e^u \right] \Big|_{u=0}^{u=\lambda} \\ &= \frac{e^{-\lambda}}{\lambda} \left(\lambda e^{\lambda} - e^{\lambda} + 1 \right) \\ &= 1 - \frac{1}{\lambda} + \frac{e^{-\lambda}}{\lambda}. \end{aligned}$$