Statistics 351 Midterm #2 – November 21, 2008

This exam is worth 50 points.

There are 4 problems on 5 numbered pages.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

You must answer all of the questions in the space provided. Note that blank space is NOT an indication of a question's difficulty.

Name: ___________________________________

Instructor: Michael Kozdron

Problem	Score
1	
2	
3	
4	

TOTAL: ____________

1. (*12 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)' \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ and that the density function of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}x_1^2 + x_1 x_2 - x_2^2\right\}, \quad -\infty < x_1, x_2 < \infty.$$

(a) Determine $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda}$.

(b) Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. Determine the distribution of the random vector $\mathbf{Y} = (Y_1, Y_2)'$.

(c) Determine the distribution of $Y_2 | Y_1 = 0$.

2. (*18 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)' \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \frac{7}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{5}{4} \end{bmatrix}.$$

(a) Determine the eigenvalues of $\boldsymbol{\Lambda}$.

(b) Find an orthogonal matrix C and a diagonal matrix D such that $CDC' = \boldsymbol{\Lambda}$.

(c) Determine $f_{\mathbf{X}}(x_1, x_2)$, the density function of $\mathbf{X}$.

(continued)

(d) Define the random vector $\mathbf{Y} = (Y_1, Y_2)'$ by setting $\mathbf{Y} = C'\mathbf{X}$ where C is the orthogonal matrix you found in **(b)**. Determine the distribution of $\mathbf{Y}$.

(e) Explain why the random variables Y_1 and Y_2 are independent.

(f) Determine the distribution of $\mathbf{X}'\mathbf{\Lambda}^{-1}\mathbf{X}$.

3. (*10 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2, X_3)'$ has a multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Find a 2×1 vector $\mathbf{a}$ such that X_2 and

$$X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$$

are independent.

4. (*10 points*) Suppose that X_1 and X_2 are independent and identically distributed with common density function

$$f(x) = a\theta^{-a}x^{a-1}, \quad 0 < x < \theta,$$

where $a > 0$ and $\theta > 0$ are parameters. As usual, let $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$, denote the minimum and maximum, respectively, of X_1 and X_2. Define the random variable

$$U = \frac{X_{(1)}}{X_{(2)}} = \frac{\min\{X_1, X_2\}}{\max\{X_1, X_2\}}$$

to be the ratio of the minimum to the maximum.

(a) Determine $f_U(u)$, the density function of U.

(b) Prove that U and $X_{(2)}$ are independent.

(c) Use **(b)** to show that $E\left(\frac{X_{(1)}}{X_{(2)}}\right) = \frac{E(X_{(1)})}{E(X_{(2)})}$.